NOTES ON LOCALIZATION IN NONCOMMUTATIVE NOETHERIAN RINGS

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Dedicated to the memory of Robert B. Warfield, Jr., 1940–1989
and to Arun V. Jategaonkar

Abstract. The purpose of these notes is to provide a brief introduction to the subject of localizing at prime ideals in noncommutative Noetherian rings. The aim is to present material connected to the concepts of links, the second layer condition, and localization at cliques of prime ideals. This is done by stating results and definitions and providing examples. Proofs are sometimes omitted or abbreviated. The bibliography lists a number of papers where these ideas are developed and used, and where the reader can find additional details.

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Introduction

These notes are a slightly modified version of my monograph of the same name, printed by the University of Granada, number 9 in their *Cuadernos de Algebra* series, in 1989. At present, the numbering of theorems, lemmas, etc., remains the same. Several references have been added, so the numbering of references has changed. I plan to make further revisions, which may result in a change in the numbering of results. Suggestions, comments, and corrections are welcome. The date of the current version is October, 1997.

In the theory of commutative rings, localization plays an extremely important role. Most basic is the idea of a quotient field, without which one cannot imagine studying integral domains. Next comes the idea of localization at a prime ideal, which transforms the given prime ideal into the unique maximal ideal and enables one to reduce many problems to the study of local rings and their maximal ideals. In these notes we will present the results of recent attempts to create analogous tools in the theory of noncommutative Noetherian rings. It has been known for some time that a naive generalization of localization at prime ideals is not possible. Recent work has shown that a reasonable theory is possible in many cases, and that it is related to the representation theory (module theory) of the ring. In fact, the ideas developed show much promise in improving our knowledge of the structure of Noetherian rings.

These notes grew out of talks I gave at the University of Utah circa 1985. Their purpose is to give examples of the situation in noncommutative rings and to state and prove the important results in localization theory. I have not attempted to provide a historical account – the reader is advised to consult the references listed in Section 8 for proper attributions of credit. I would like to express my gratitude to the mathematicians who have contributed to the development of this subject for their interesting work and to several of them for helpful conversations on this material.

In Section 1 we review some generalities about noncommutative localization and Noetherian rings, and give Goldie's very satisfactory answer (from about 1958) to the quotient ring problem, which shows that for Noetherian rings one can operate more or less as in the commutative case. In the rest of the notes we consider the question of localizing at a prime ideal, and here the answer is much more complicated, and not yet complete. (Most of this work has been done since 1980.) In Section 2 we give some examples illustrating the obstructions to localization that arise in noncommutative rings. One of these is that a single prime ideal may be linked to other prime ideals, and we give examples of the kind of links that can occur; the other is that the second layer condition may fail. Both of these conditions involve the existence of special extensions of modules related to the prime ideal in question, and links also involve connections between the right and left structure of the ring. In Section 3 we note that if prime ideals satisfy the Artin-Rees Lemma these obstructions vanish. Section 4 explores in more detail the connection between the obstructions and module extensions, including results about rings where the second layer condition holds or does not hold. In it we prove Jategaonkar's Main Lemma along with other useful technical results. In Section 5, we use these results to give necessary and sufficient conditions for the existence of a generalized localization at a prime ideal (gotten by inverting the elements which are regular modulo all of the prime ideals joined by links to the original one – we call this the localization at the clique of the prime). The main result is that the clique of a prime ideal is classically localizable if and only if it satisfies the second layer condition and the intersection condition. We also state some

better results for p.i. rings. In Section 6 we begin by showing that any finite set of prime ideals closed under links and satisfying the second layer condition can be localized in the above sense. We then topologize the set of prime ideals, which enables us to prove a prime ideal is linked to only countably many others and to prove more specific results about localization which apply to some classes of rings of current interest, such as group rings, enveloping algebras, and rings which satisfy a polynomial identity. In Section 7, we study rings with the second layer condition in a little more detail, and we prove a symmetry result about bimodules over such rings (which shows that in such rings there are no unequal comparable primes joined by links). We also give some results on finite ring extensions. In Section 8 we discuss references and other work. Two appendices are included: the first summarizes some of the known results about localization in enveloping algebras and their generalizations. The second gives an outline of the known results about localization in group rings and then sketches the proof that rings strongly graded by a polycyclic-by-finite group over a commutative Noetherian base ring satisfy the second layer condition.

Conventions and definitions

The symbol " $\subset$ " indicates proper containment. Generally R is a ring (always with identity) and modules are (unital) right modules; we use R_R to indicate R regarded as a right R-module with ring multiplication as action. If M is an R-module, we denote its injective hull by $E(M)$. An element of a ring is *regular* if it is not a zero divisor. If I is an ideal of R, an element $r \in R$ is *regular modulo* I if $r + I$ is regular in R/I: we denote the set of all such elements by $\mathcal{C}_R(I)$ or simply $\mathcal{C}(I)$. If A and B are sets for which these make sense, we put $\text{r-ann}_B(A) = \{\, b \in B \mid Ab = 0 \,\}$, $\text{l-ann}_B(A) = \{\, b \in B \mid bA = 0 \,\}$: when $B = R$ we omit it, and when only one of "r-ann" or "l-ann" makes sense, we write "ann" instead. Properties which are mentioned without a side and which could hold on either side are assumed to hold on both sides; for example, "ideal" means "right and left ideal", "Noetherian ring" means "right and left Noetherian ring", and "classically localizable" means "classically right and classically left localizable". In a noncommutative ring, an ideal is *prime* if whenever it contains a product of ideals, it contains one of the ideals, and *semiprime* if whenever it contains a power of an ideal, it contains that ideal. When the usual (stronger) commutative property holds, that is, when we can replace ideals by elements in the definition, we say the ideal is *completely prime* or *completely semiprime*. When 0 is a [semi]prime ideal, we say R is a *[semi]prime ring*. A subset of a ring is *nil* if every element of it is nilpotent; the set is *nilpotent* if there is a fixed n such that any product of n elements from the set is 0. If $\mathcal{X}$ is a collection of sets, $\cap\mathcal{X} = \cap_{X\in\mathcal{X}} X$.

1. Quotient rings and Goldie's Theorem

In this section we give a quick review of some of the basics of localization theory and the theory of Noetherian rings. Many of the proofs are deliberately sketchy. See Section 8 for references to some more detailed treatments.

Given a ring R and a subset C, a *universal C-inverting ring* is a ring S together with a ring map $i : R \to S$ such that (a) $i(c)$ is a unit of S for all $c \in C$ and (b) for any ring S' and ring map $i' : R \to S'$ such that $i'(c)$ is a unit of S' for all $c \in C$,

there is a unique ring map $f : S \to S'$ such that $fi = i'$. One can show such an S and i exist, and of course they are unique up to isomorphism. Unfortunately, the ring S is not easy to write down in an explicit way. What would be more useful would be to be able to form a ring of fractions with denominators from C.

Suppose C is a candidate for a set of denominators, i.e., suppose we want to form a ring of fractions of the form r/c where $r \in R$ and $c \in C$. Clearly the set C should be multiplicatively closed (and for convenience we always assume $1 \in C$), and this is enough for a ring of fractions to exist in a commutative ring R. For a noncommutative ring, we first have to decide whether r/c means rc^{-1} or $c^{-1}r$, as different answers arise from these two choices. A *right localization of R with respect to C* is a ring S and ring map $i : R \to S$ such that (i) $i(c)$ is a unit in S for each $c \in C$, (ii) every element of S has the form $i(r)i(c)^{-1}$ for some $r \in R, c \in C$, and (iii) $i(r)i(c)^{-1} = i(r')i(c)^{-1}$ if and only if $rd = r'd$ for some $d \in C$. It is not hard to see that such an S is a universal C-inverting ring, and so is unique; thus we can safely denote S by RC^{-1} when it exists. We will suppress the map i and write the elements of RC^{-1} as rc^{-1} (so $i(r) = r1^{-1}$). Condition (iii) implies that the kernel of the map i is $\{ r \in R \mid rc = 0 \text{ for some } c \in C \}$ — we call the elements of this set the *C-torsion* elements of R. Hence i is injective if C consists of regular elements, and in this case we do not need the d in condition (iii).

If such a localization exists, we can write $c^{-1}r$ in the canonical form, and so $c^{-1}r1^{-1} = r_1c_1^{-1}$ for some $r_1 \in R, c_1 \in C$. Hence $rc_1 1^{-1} = cr_1 1^{-1}$ and so $rc_1c_2 = cr_1c_2$ for some $c_2 \in C$. Thus if $s = r_1c_2$ and $d = c_1c_2$, we have $rd = cs$ and $d \in C$. If for all $r \in R$ and $c \in C$ such s, d exist, we say C satisfies the *right Ore condition*: we have just shown this condition must hold for RC^{-1} to exist.

Suppose $r \in R$ and $c \in C$ and $cr = 0$ in R. Then in RC^{-1}, we have $0 = 01^{-1} = c^{-1}01^{-1} = c^{-1}cr1^{-1} = r1^{-1}$, so $rd = 0$ for some $d \in C$. If such a d always exists, we say C is *right reversible*. Again, we have shown this is necessary for RC^{-1} to exist.

Theorem 1.1 (Øre)**.** *Let C be a multiplicatively closed set in R. The right localization RC^{-1} exists if and only if C is a right Ore, right reversible subset of R. Moreover, RC^{-1} is a universal C-inverting ring, and so is unique up to a natural isomorphism.*

Remark . Such a subset C of R is called a *right denominator set*.

To prove this theorem, it is handy to first prove the following lemma. Given any multiplicatively closed subset C of R and a right R-module M, we say an element $m \in M$ is *C-torsion* if $mc = 0$ for some $c \in C$, and we say M is *C-torsion* if every element of M is C-torsion. Generally one says M is *C-torsionfree* if no nonzero submodule of M is C-torsion; when C is a right Ore set, the next lemma implies that this is true precisely when 0 is the only C-torsion element of M, and that is the characterization we will use in these notes. When C is the set of regular elements of R, we speak simply of *torsion*.

Lemma 1.2. *Let C be a multiplicatively closed set in R.*

(a) (Right common multiple property) If C is a right Ore set in R and $c_1, \ldots, c_n \in C$, there are $r_1, \ldots, r_n \in R$ such that $c_1r_1 = \cdots = c_nr_n \in C$. (Thus any finite collection of fractions in RC^{-1} can be simultaneously brought to a common denominator.)

*(b) The set C is right Ore if and only if for every right R-module M, the subset $T = \{ m \in M \mid mc = 0 \text{ for some } c \in C \}$ is a submodule (called the C-*torsion submodule*). If $M = R$ and C is right Ore, then T is an ideal.* ∎

Just as in commutative ring theory, one can prove Theorem 1.1 by defining RC^{-1} to be a set of equivalence classes in $R \times C$ and tediously verifying the ring axioms. Alternatively, one can realize the localization as a subfactor ring of the endomorphism ring of the injective hull of (a factor ring of) R. Suppose C consists of regular elements (we can always reduce to this case by factoring out the ideal of C-torsion elements of R) and set $E = E(R_R)$, $F = \mathrm{End}_R(E)$, $I = \{ f \in F \mid f(R) = 0 \}$, and $T = \{ f \in F \mid If \subseteq I \}$. ($T$ is thus the *idealizer* $\mathbb{I}_F(I)$ of I in F, that is, the largest subring of F in which the one-sided ideal I is a two-sided ideal.) Define $\phi : R \to F$ by defining $\phi(x)$ to be some extension of the map $r \mapsto xr$. Then $\phi(x) \in T$, so ϕ induces a map $i : R \to T/I$, which is well-defined (that is, all the original choices made in defining ϕ are now irrelevant). Furthermore, each $i(c)$ is a unit in T/I The subring of T/I generated by $i(R) \cup \{ i(c)^{-1} \mid c \in C \}$ satisfies the conditions required of RC^{-1}.

Of course one can define the analogous conditions on the left and get the notions of a *left localization* and a *left denominator set*. It is clear from the universal C-inverting property of localizations that if C is both a right and left denominator set, one can naturally identify the localizations RC^{-1} and $C^{-1}R$.

One can also define the notion of a quotient module MC^{-1} over RC^{-1} for a right R-module M, with elements of the form mc^{-1} and with the kernel of the map $m \mapsto m1^{-1}$ being the C-torsion submodule of M. Instead of constructing it as above, though, we can use $M \otimes_R RC^{-1}$ as MC^{-1}. For example, if $R = \mathbb{Z}$ and $C = \mathbb{Z} \setminus \{0\}$, then $MC^{-1} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is just a $\mathbb{Q}$-vector-space of dimension equal to the torsionfree rank of the Abelian group M.

Lemma 1.3. *If C is a right denominator set in R, then ${}_RRC^{-1}$ is flat, and localization of right R-modules induces an exact functor.* ∎

If we let R be a domain and $C = R \setminus \{0\}$, we get the following result.

Proposition 1.4 (Øre)**.** *(a) A domain R has a right division ring of fractions if and only if any two nonzero right ideals have nonzero intersection.*

(b) Any right Noetherian domain has a right division ring of fractions. ∎

Example 1.5. The ring $R = k\{x, y\}$, where k is a field and x and y are noncommuting indeterminates, has neither a right nor a left division ring of fractions since $xr = ys$ or $rx = sy$ only if $r = s = 0$. (R can, however, be embedded in a division ring — in the quotient division ring $D_1(k)$ of the first Weyl algebra $A_1(k)$ if k has characteristic 0.)

In commutative ring theory, many results depend on chain conditions. This is even more true in the noncommutative theory, as the contrast between Example 1.5 and Proposition 1.4 shows. Thus from now on we will assume all rings in question are at least right Noetherian. This leads to a simplification of the requirements for a right denominator set, as the next lemma shows. Note also that it is clear that the right Ore condition remains valid in factor rings, but this is not clear for reversibility. The next result shows that for right Noetherian rings, everything is O.K.

Lemma 1.6. *If C is a multiplicatively closed right Ore set in the right Noetherian ring R, then C is right reversible.*

Proof. Suppose $cr = 0$. Using right Noetherianness, we may suppose r-ann(c) = r-ann(c^2). By the right Ore condition, $cs = rd$ for some $s \in R$ and $d \in C$. Then $c^2 s = crd = 0$, so $cs = 0$. Thus $rd = 0$. ∎

Of course Proposition 1.4 is important, but many nice rings like the ring of 2×2 matrices over the integers are not domains. Domains are simply much less common in the noncommutative setting – it is more natural to consider prime (and occasionally semiprime) rings. Furthermore, fields are replaced by simple Artinian rings as building blocks in noncommutative ring theory. Thus we need a stronger result, and this was given by A. W. Goldie in 1958–1960, who proved that if R is a prime right Noetherian ring, then the set C of regular elements is a right denominator set and the localization RC^{-1} is a simple Artinian ring. One of the key points in the proof of Goldie's Theorem is that in a semiprime right Noetherian ring, every essential right ideal contains a regular element, and this turns out to be a very useful technical result in its own right. The proof we give is not the original one, but its ideas are due mainly to Goldie.

Recall that a submodule of a module is *essential* if it has nonzero intersection with every nonzero submodule. A right ideal is called essential if it is an essential submodule of R_R. A module is *uniform* if every nonzero submodule is essential. Thus Proposition 1.4 says a domain R has a right division ring of fractions if and only if R_R is uniform. The *uniform dimension* (also called *Goldie dimension* or *uniform rank* or *Goldie rank*) of a module is the maximum size of a set of nonzero submodules whose sum is direct, or equivalently, the number of uniform submodules in any set whose sum is direct and an essential submodule. For a finitely generated right module M over a right Noetherian ring R, the uniform dimension is always finite, and the uniform dimension of a submodule is less than or equal to that of M, with equality if and only if the submodule is essential. When M is C-torsionfree for a right denominator set C, essential and uniform submodules remain so after localization, and the uniform dimension is not changed. For any right R-module M, the *singular submodule* of M is the set of elements of M whose annihilator is an essential right ideal. The *right singular ideal* of R is the singular submodule of R_R, and it is an ideal.

Lemma 1.7. *If R is a right Noetherian ring, the prime radical (the intersection of all the prime ideals of R) and the right singular ideal are nilpotent. Moreover, the prime radical contains every nil one-sided ideal.*

Proof. If Z is the right singular ideal of R, choose an n with $\text{r-ann}(Z^n) = \text{r-ann}(Z^{n+1})$ and suppose $Z^{n+1} \neq 0$. Choose $z \in Z$ with $Z^n z \neq 0$ and $\text{r-ann}(z)$ as big as possible. Using the definition of Z, one sees that $\text{r-ann}(xz) \supset \text{r-ann}(z)$ for all $x \in Z$, which implies by our choice of z that $Z^n xz = 0$. Hence $Z^{n+1}z = 0$, contrary to our choice of n. Thus $Z^{n+1} = 0$.

Let K be a maximal nilpotent ideal of R. Clearly K is contained in the prime radical of R, and so we may pass to R/K. Thus we can assume R has no nonzero nilpotent ideals. We note that the prime radical is always a nil ideal, since if x is a nonnilpotent element of R, any ideal maximal with respect to not containing any power of x is prime, whence x is not in the prime radical of R. Thus to complete the proof, we need to show every nil one-sided ideal of R is zero.

Let I be a nonzero left ideal of R consisting of nilpotent elements and choose a nonzero $y \in I$ with $\text{r-ann}(y)$ as big as possible. Replace I by Ry and note that $y^2 = 0$. We will show $I^2 = 0$, contradicting the fact that $I \neq 0$. Suppose $r \in R$. Either $ry = 0$ or $\text{r-ann}(ry) = \text{r-ann}(y)$. In the latter case, $(ry)^n = 0$ but $(ry)^{n-1} \neq 0$ for some $n \geq 2$. Then we see $y(ry)^{n-2} \neq 0$ but $(yry)(ry)^{n-2} = y(ry)^{n-1} = 0$, and so maximality

of r-ann(y) implies $yry = 0$. Thus for all $r \in R$, we have $yry = 0$, which implies $I^2 = RyRy = 0$.

Since for any $r, x \in R$, it is the case that rx is nilpotent if and only if xr is nilpotent, the existence of a nonzero one-sided nil ideal of R implies the existence of a left ideal I as above. This proves the lemma. ∎

Lemma 1.8. *If R is a semiprime right Noetherian ring of right uniform dimension n, then no chain of right annihilators in R can have length greater than n. In particular, R has the d.c.c. on right annihilators.*

Proof. First we show that if A and B are right ideals of R such that A is a right annihilator and A is an essential submodule of B, then $A = B$. Suppose $A =$ r-ann(X) for some subset X of R and $b \in B$. Then one can show that $J = \{r \in R : br \in A\}$ is an essential right ideal of R. Now $bJ \subseteq A$, so $XbJ = 0$. By the last lemma, this implies $Xb = 0$, so $b \in$ r-ann$(X) = A$, proving the claim.

Since R has right uniform dimension n, one can show that any chain of right ideals of R such that no member of the chain is an essential submodule of the next larger element of the chain has length at most n. (If A is a nonessential submodule of B, then there is a nonzero submodule C of B with $A \cap C = 0$.) Combining this fact with the result of the last paragraph completes the proof of the lemma. ∎

Theorem 1.9 (Key to Goldie's Theorem)**.** *If R is a semiprime right Noetherian ring, then a right ideal of R is essential if and only if it contains a regular element.*

Proof. First of all we show that if a is an element of R such that r-ann$(a) =$ r-ann(a^2), then $aR \cap$ r-ann$(a) = 0$ and $aR +$ r-ann(a) is an essential right ideal of R. (Note that if x is any element of R, then some power of x has this property.) That the intersection is 0 is clear. Suppose I is a right ideal of R such that $I \cap (aR +$ r-ann$(a)) = 0$. Then it is easy to see that the sum $I + aI + a^2I + \cdots$ is direct, violating Noetherianness unless some $a^kI = 0$. This implies $I \subseteq$ r-ann(a), so $I = 0$.

($\Leftarrow$) This follows immediately from the above. There is an alternative proof that is also worth noting. Let c be a right regular element of R. Then the map $r \mapsto cr$ is an injective R-module homomorphism, so the right ideal cR is isomorphic as an R-module to R_R. It therefore has the same uniform dimension, and hence it is an essential submodule.

($\Rightarrow$) Let I be an essential right ideal of R. By Lemma 1.8, we can find an element $a \in I$ with r-ann(a) as small as possible subject to the condition r-ann$(a) =$ r-ann(a^2). We will show aR is an essential right ideal of R. If this is not so, then the first paragraph implies r-ann$(a) \neq 0$, and so $I \cap$ r-ann(a) is a right ideal, which by Lemma 1.7 is not nil. Thus there is a nonzero $b \in I \cap$ r-ann(a) such that $aR \cap bR = 0$ and r-ann$(b) =$ r-ann(b^2). For this b, we have r-ann$((a+b)^2) =$ r-ann$(a+b) =$ r-ann$(a) \cap$ r-ann(b); by minimality this implies r-ann$(a) \subseteq$ r-ann(b). Since $b \in$ r-ann(a), this implies that $b^2 = 0$, which contradicts our choice of b.

Hence aR is an essential right ideal of R and the first paragraph implies that r-ann$(a) = 0$. Furthermore, l-ann$(a) =$ l-ann(aR), so Lemma 1.7 shows that l-ann$(a) = 0$. Thus a is a regular element of I. ∎

Remark . Theorem 1.9 implies that every nonzero ideal in a prime right Noetherian ring contains a regular element, since such an ideal is essential as a right ideal.

Theorem 1.10 (Goldie)**.** *Suppose R is a right Noetherian ring and C is the set of regular elements of R. Then the localization RC^{-1} exists (i.e., C is a right denominator set) and is semisimple Artinian if and only if R is semiprime. In this case, RC^{-1} is simple if and only if R is prime.*

Proof. First, suppose R is a semiprime right Noetherian ring and C is the set of its regular elements. If $c \in C$ and $r \in R$, then since cR is an essential right ideal of R, one can show that $J = \{ x \in R \mid rx \in cR \}$ is an essential right ideal of R. This implies J contains an element of C, which in turn clearly implies that C is a right Ore set.

Since C is a set of regular elements, it is not hard to see that essential right ideals of RC^{-1} are localizations of essential right ideals of R. But every essential right ideal of R contains an element of C and so localizes to all of RC^{-1}. This means there are no proper essential right ideals in RC^{-1}; it is well-known that this implies RC^{-1} is a semisimple Artinian ring.

Suppose now that RC^{-1} exists. One can show in this situation that R is [semi]prime if and only if RC^{-1} is [semi]prime, using for example Lemma 2.1(b). (In fact, the minimal primes of R localize to the minimal primes of RC^{-1}.) Since an Artinian ring is [semi]prime if and only if it is [semi]simple, this completes the proof. ∎

Since C consists of regular elements, we may regard R as a subring of RC^{-1}: we call RC^{-1} the *right Goldie quotient ring* of R. If R is a left and right Noetherian prime ring, then the set C of regular elements of R is a right and left denominator set and $RC^{-1} = C^{-1}R$ is a simple Artinian ring. (Note that this shows that the uniform dimension of R is the same on the left and the right in this case, since it is the unique integer n for which the Goldie quotient ring is isomorphic to a ring of $n \times n$ matrices over a division ring.)

One of the consequences of Theorem 1.9 we will often use is that an element in a semiprime right Noetherian ring whose right annihilator is 0 is actually regular (i.e., its left annihilator is also 0). (See the last paragraph of the proof of Theorem 1.9.) There are many other facts and technical tools which grow out of Goldie's Theorem and its proof. As an example, we now state without proof a pair of results by Stafford and Small.

Lemma 1.11. *Let R be a right Noetherian ring, I a right ideal of R, and $a \in R$.*

(a) If X is a finite set of prime ideals of R such that $(aR + I) \cap \mathcal{C}(P)$ is nonempty for all $P \in X$, then there is a $c \in I$ such that $a + c \in \bigcap\{ \mathcal{C}(P) \mid P \in X \}$.
Note: if no two primes in X are comparable, $\bigcap\{ \mathcal{C}(P) \mid P \in X \} = \mathcal{C}(\cap X)$.

(b) If R is Noetherian and $aR + I$ contains a regular element, then there is a $c \in I$ such that $a + c$ is regular. ∎

Theorem 1.12 (Small's Theorem)**.** *Let R be a right Noetherian ring with prime radical N. Then $\mathcal{C}(0) \subseteq \mathcal{C}(N)$, and $\mathcal{C}(0) = \mathcal{C}(N)$ if and only if $\mathcal{C}(0)$ is a right denominator set and $R\mathcal{C}(0)^{-1}$ is right Artinian.* ∎

Some comments: in the ring R of 2×2 matrices over the integers, C is the set of all matrices with nonzero determinant. But clearly the Goldie quotient ring of R is the ring of 2×2 matrices over the rationals, and we need only invert nonzero scalar matrices to get it. Note that if $x, y \in R$, then $(xy - yx)^2$ is a scalar matrix and so $(xy - yx)^2 z - z(xy - yx)^2 = 0$ for all $x, y, z \in R$. A ring R is called a *p.i. ring* if there is a polynomial $p(x_1, \ldots, x_n)$ in noncommuting indeterminates with coefficients in

the center of R such that $p(r_1, \ldots, r_n) = 0$ for all choices of $r_1, \ldots, r_n \in R$. (To avoid triviality, we need to make some assumptions about the coefficients of p, say that some coefficient is a unit. We can actually get all coefficients to be ± 1.) In a prime p.i. ring or a right Noetherian semiprime p.i. ring, every essential right ideal contains a regular central element, so we can get to the Goldie quotient ring by inverting only central regular elements, yielding of course a left and right localization all at once. (In fact, a semiprime right Noetherian p.i. ring is also left Noetherian.) If R is prime, the resulting quotient ring is a finite dimensional algebra over its center.

We define a right Noetherian ring to be *right FBN (fully bounded Noetherian)* if in every prime factor ring, every essential right ideal contains a nonzero ideal. This is equivalent to the assumption that over any prime factor ring, a finitely generated right module is faithful if and only if it is torsionfree (with respect to the set of regular elements). It can also be shown to be equivalent to the assumption that the annihilator of any finitely generated right module is the annihilator of some finite subset. The existence of central elements mentioned above shows a right Noetherian p.i. ring is right FBN.

As defined above, an element of a module is called singular if its annihilator is an essential right ideal of R, and that the set of singular elements of any module is a submodule. Goldie's Theorem implies that if R is a semiprime right Noetherian ring, "torsion" and "singular" are synonyms. Over such a ring, an essential extension of a torsionfree module is torsionfree, and hence a uniform module is either torsion or torsionfree, while the quotient of a module by an essential submodule is torsion.

A very important fact we will use often is this: if R is a semiprime right Noetherian ring and M is a nonzero torsionfree right R-module (or just a nontorsion right R-module), then M contains a copy of a nonzero right ideal of R. If R is also left Noetherian, then every finitely generated torsionfree right R-module M can be embedded in a finitely generated free right R-module; if M is uniform, then it can be embedded in R_R. (This last result can be proved using the fact that the left Goldie quotient ring $\mathcal{C}(0)^{-1}R$ "equals" $R\mathcal{C}(0)^{-1}$ and the fact that every finitely generated torsionfree right R-module embeds in a finite direct sum of copies of $R\mathcal{C}(0)^{-1}$.)

2. Localizing at prime ideals

The next item on our agenda is localizing at a prime ideal. Following Goldie's Theorem, and because we want to preserve as much information as possible about P, we say a prime ideal P in R is *right localizable* if $\mathcal{C}(P) = \{ r \in R \mid r + P \text{ is regular in } R/P \}$ is a right denominator set in R. In this case we denote $R\mathcal{C}(P)^{-1}$ by R_P. Before we discuss when we can localize at P, we need to discuss some generalities; we also examine the properties of the localization R_P.

First of all we note some technical results.

Lemma 2.1. *Suppose R is a right Noetherian ring, C is a right denominator set in R, and I is an ideal of R.*

(a) RC^{-1} is a right Noetherian ring.

(b) $IC^{-1} = IRC^{-1}$ is an ideal of RC^{-1} and if $\bar{R} = R/I$ and $\bar{C} = \{ c + I \mid c \in C \}$, then $\bar{C}$ is a right denominator set in $\bar{R}$ and $\bar{R}\bar{C}^{-1}$ is isomorphic to RC^{-1}/IC^{-1}.

(c) If I is prime and disjoint from C, then $C \subseteq \mathcal{C}(I)$ and IC^{-1} is prime in RC^{-1}.

(d) If R is p.i. or right FBN, so is RC^{-1}.

Proof. The proof is routine: we just give two examples of the techniques.

(b) Suppose I is an ideal of R. Then we have a chain $IC^{-1} \subseteq c^{-1}IC^{-1} \subseteq c^{-2}IC^{-1} \subseteq \cdots$ for any $c \in C$. Since RC^{-1} is right Noetherian, there is an n with $c^{-n}IC^{-1} = c^{-(n+1)}IC^{-1}$. Thus $IC^{-1} = c^{-1}IC^{-1}$. Since this is true for any $c \in C$, we see $RC^{-1}IC^{-1} \subseteq IC^{-1}$, which shows IC^{-1} is an ideal of RC^{-1}.

(c) Now suppose I is prime and disjoint from C. We may pass to R/I and assume R is prime right Noetherian, $I = 0$, and $0 \notin C$. Let $T = \{ r \in R \mid rc = 0 \text{ for some } c \in C \}$. Using the common multiple property, one can show that T is an ideal of R. If $T \neq 0$, then T contains a regular element of R, violating the definition of T. Thus $T = 0$ and C consists of left regular elements. By reversibility, it consists of regular elements, and so $C \subseteq \mathcal{C}(I)$. ∎

Corollary 2.2. *Let R be a right Noetherian ring and C be a right denominator set in R. Then there is a bijective order-preserving correspondence between primes of R disjoint from C and primes of RC^{-1} given by $P \mapsto PC^{-1}$. If $C \subseteq \mathcal{C}(0)$, then the inverse is given by $Q \mapsto Q \cap R$.* ∎

Now suppose R is right Noetherian and P is a right localizable prime ideal of R. Then Corollary 2.2 shows that there is a one-to-one correspondence between primes of R contained in P and primes of R_P. In particular, PR_P is the unique maximal ideal of R_P. By Goldie's Theorem and Lemma 2.1, R_P/PR_P is isomorphic to the Goldie quotient ring of R/P, and so is Artinian.

We can in fact show that $PR_P = J(R_P)$. Set $C = \mathcal{C}(P)$ and suppose S is a simple right R_P-module, $s \in S$, and $sP \neq 0$. Then $S = sPR_P = sPC^{-1}$, so for some $p \in P, c \in C$, $s = spc^{-1}$ and hence $s(p-c)1^{-1} = 0$. But $p - c \in C$, so we must have $s = 0$. This contradiction shows that $SP = 0$ and so the annihilator of S in R_P is PR_P. Note that $J(R_P)^n = P^nR_P$ for any n and that each ring $R_P/J(R_P)^n$ is Artinian.

If R is commutative Noetherian (or right FBN), any finitely generated right R_P-module M containing the simple module S as an essential submodule is Artinian, or equivalently, $MP^n = 0$ for some n. If this holds in the noncommutative case for a right localizable prime ideal P of R, we say P is *classically right localizable*. One can show that this holds if and only if $J(R_P)$ has the right AR property (see Section 3 for the definition). It is an open question whether right localizability implies classical right localizability for a prime ideal in a Noetherian ring.

Can we prove a "Goldie's Theorem" for localizing at prime ideals? The answer is no: let's see why not.

Example 2.3. Let k be a field and let R be the ring of 2×2 upper triangular matrices over k. The ring R is an Artinian ring with two prime ideals, the ideal Q of matrices in R whose upper left corner is 0 and the ideal P of matrices in R whose lower right corner is 0. Note that R/P and R/Q are both isomorphic to k, and that $QP = 0$ while $PQ = P \cap Q = J(R)$. Also note that $\mathcal{C}(Q) = R \setminus Q$ and $\mathcal{C}(P) = R \setminus P$. For $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{C}(Q)$ and $\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in R$, we have $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a^{-1}d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so Q is right localizable. Note that the $\mathcal{C}(Q)$-torsion is Q, so $R_Q \cong R/Q \cong$

k. On the other hand, $\mathcal{C}(P)$ is not right Ore since $\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}a&b\\0&c\end{pmatrix}=\begin{pmatrix}0&0\\0&c\end{pmatrix}$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}d&e\\0&f\end{pmatrix}=\begin{pmatrix}0&f\\0&0\end{pmatrix}$, and these can never be equal for $f\neq 0$.

Why does this happen? Note that $QJ=JP=0$, and that J is faithful and torsionfree as both a right R/P- and left R/Q-module.

Lemma 2.4. *Suppose R is a right Noetherian ring, C is a right denominator set in R, and A and B are ideals of R with $A\subset B$. Suppose also that P and Q are prime ideals of R with* r-ann$(B/A)=P$ *and* l-ann$(B/A)=Q$ *and that B/A is torsionfree as a right R/P-module. If $C\subseteq\mathcal{C}(P)$, then also $C\subseteq\mathcal{C}(Q)$.*

Proof. Suppose $c\in C\setminus\mathcal{C}(Q)$. Goldie's Theorem implies there is an $r\in R\setminus Q$ with $cr\in Q$. Now $cr(B/A)=0$, so for each $b\in B$, we have $\bar{c}\bar{r}\bar{b}=\bar{0}$ in $\bar{R}=R/A$, where $\bar{}$ denotes the natural map. Since $\bar{C}$ is right reversible in $\bar{R}$, there is a $d\in C$ with $\bar{r}\bar{b}\bar{d}=\bar{0}$. But B/A is C-torsionfree, so $\bar{r}\bar{b}=\bar{0}$, and we conclude that $r(B/A)=0$. This contradicts the fact that l-ann$(B/A)=Q$. ∎

Thus to localize the ring R of Example 2.3 at a right Ore set inside $\mathcal{C}(P)$, we must include $\mathcal{C}(Q)$. Since $\mathcal{C}(Q)\cap\mathcal{C}(P)=$ units(R), we can't localize R any further: R is already "local".

There is another connection between Q and P in Example 2.3. The ring R has exactly two simple right modules: J_R, which is isomorphic to R/P, and $(Q/J)_R$, which is isomorphic to R/Q. The only nonsplit extension of simple right R-modules is given by the exact sequence $0\to J\to Q\to Q/J\to 0$. We have the following analogue of Lemma 2.4.

Lemma 2.5. *Suppose R is a right Noetherian ring, C is a right Ore set in R, and P and Q are prime ideals of R. Suppose there is an exact sequence $0\to L\to M\to N\to 0$ of uniform right R-modules such that* ann$(L)=P$, ann$(N)=Q$, $L=\text{ann}_M(P)$, *and L is torsionfree as a right R/P-module. If $C\subseteq\mathcal{C}(P)$, then also $C\subseteq\mathcal{C}(Q)$.* ∎

Proof. Suppose the conclusion fails. Then there is a $c\in C\cap Q$, whence $Mc\subseteq L$. This implies $McP=0$. Since $c\in\mathcal{C}(P)$ and C is right Ore, this implies MP is C-torsion. However, L is C-torsionfree and essential in M, so this can only happen if $MP=0$. This contradiction completes the proof. ∎

We will see in Lemma 4.2 that the connection between Q and P via $Q\cap P$ mentioned above and the connection via extensions of uniform right ideals of R/P and R/Q are equivalent, and so we will concentrate on the first type of connection for the time being.

The theory we now aim to be build can be built, at least partially, for right Noetherian rings (see comments below), but it is more convenient to work with a (right and left) Noetherian rings, and we will usually work in this setting. If R is a Noetherian ring and Q,P are prime ideals in R, we say Q is *ideal-linked* to P (via $A\subset B$) if there are ideals A,B of R with $A\subset B$ such that l-ann$(B/A)=Q$ and r-ann$(B/A)=P$ and B/A is torsionfree as both a right R/P- and left R/Q-module. When R is Noetherian, the torsionfreeness condition is equivalent to the condition that every nonzero subbimodule of B/A is faithful as both a right R/P- and left R/Q-module, as can be seen by considering the annihilators of the appropriate torsion submodules, noting

that the torsion submodules are actually subbimodules and hence have their annihilators determined by a finite subset. In case R is only right Noetherian, these conditions may not be equivalent – in fact, it may not make sense to speak of R/Q-torsion on the left. In this case we alter the definition of a link as follows: instead of requiring torsionfreeness on the left over R/Q, we require that every nonzero subbimodule of B/A be faithful on the left over R/Q. Now Lemma 2.4 works for any A, B, but it turns out that we are mainly interested in a special A and B. We say Q is *linked* to P (via $A \subset Q \cap P$), denoted $Q \rightsquigarrow P$, if we can choose $B = Q \cap P$ and $QP \subseteq A \subset Q \cap P$ in the above. (When we want to distinguish more clearly, we call this a *second layer link* or a *direct link*.)

If Q and P are maximal ideals, it is not hard to see that $Q \rightsquigarrow P$ if and only if $QP \subset Q \cap P$. The more general question of whether two prime ideals Q and P in a Noetherian ring R are linked whenever $\text{l-ann}(Q \cap P/QP) = Q$ and $\text{r-ann}(Q \cap P/QP) = P$ is still open. (This is true if $Q \cap P/QP$ is torsionfree as a right R/P-module, although the link need not be via QP. It is true without the torsionfreeness assumption when all the prime ideals in R satisfy the right and left second layer conditions, conditions which we will define soon.)

We say a subset X of the set $\operatorname{Spec} R$ of prime ideals of R is *right link-closed* (or *right stable*) if whenever $P \in X$, $Q \in \operatorname{Spec} R$, and $Q \rightsquigarrow P$, we have $Q \in X$. We say $X \subseteq \operatorname{Spec} R$ is *link-closed* or *stable* if $Q \rightsquigarrow P$ implies either both $Q, P \in X$ or both $Q, P \notin X$. If $P \in \operatorname{Spec} R$, the *right clique of* P, denoted $\operatorname{rt\,cl}(P)$, is the smallest right link-closed subset of $\operatorname{Spec} R$ containing P; the *clique of* P, $\operatorname{cl}(P)$, is the smallest such link-closed subset. Thus we have:

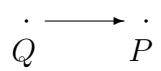
Corollary 2.6. *If R is a right Noetherian ring, P is a prime ideal in R, and C is a right Ore set in R disjoint from P, then $C \subseteq \bigcap\{\, \mathcal{C}(Q) \mid Q \in \operatorname{rt\,cl}(P) \,\}$. If C is an Ore set and R is Noetherian, then $C \subseteq \bigcap\{\, \mathcal{C}(Q) \mid Q \in \operatorname{cl}(P) \,\}$.* ∎

G. Sigurdsson has given an example of a Noetherian ring containing a prime ideal P such that the clique of P is strictly bigger than the union of the left and right cliques of P.

For future reference we note a few things about links between prime ideals in Noetherian rings. First, we note that $Q \rightsquigarrow P$ in R if and only if $Q/QP \rightsquigarrow P/QP$ in R/QP. Second, if C is a right denominator set in R disjoint from Q and P, then $Q \rightsquigarrow P$ in R if and only if $QC^{-1} \rightsquigarrow PC^{-1}$ in RC^{-1}. This latter fact implies that a right link-closed set of primes in $\operatorname{Spec} R$ disjoint from C yields a right link-closed set in $\operatorname{Spec} RC^{-1}$ after localization. Third, $Q \rightsquigarrow P$ in R if and only if $P \rightsquigarrow Q$ in R^{op}. Fourth, if $Q \rightsquigarrow P$, then Q and P have the same intersection with the center of R. Finally, we recall a result of Lenagan's which states that if B is an R-R-bimodule which is Noetherian as both a right and a left R-module, then $R/\,\text{l-ann}(B)$ is a left Artinian ring if and only if $R/\,\text{r-ann}(B)$ is a right Artinian ring, and both of these conditions are equivalent to the condition that B is Artinian as either a right or left R-module. Thus if Q, P are prime ideals of R with $Q \rightsquigarrow P$, we see that R/Q is an Artinian ring if and only if R/P is an Artinian ring.

We can draw a graph of $\operatorname{Spec} R$ with arrows indicating links; we call this graph the *link graph of* R.

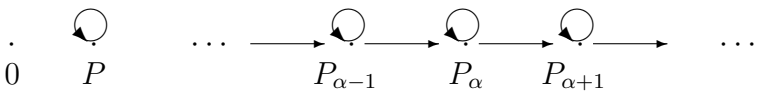
Example 2.7. (a) The ring R of Example 2.3 has link graph



(b) If R is a commutative Noetherian domain, its link graph is

(c) Any finite directed graph can be realized as the link graph of a finite-dimensional k-algebra, where k is an arbitrary field. We remark that given any Artinian algebra there is a finite directed graph associated to it called its quiver. The vertices of the quiver can be labelled in various equivalent ways, one of which is by the prime ideals (= annihilators of the simple modules) of the algebra, and when they are so labelled, Jategaonkar proves that the quiver is the directed graph of the ideal links in the algebra. There is a standard way of producing a finite-dimensional algebra whose quiver is any given finite directed graph, and for this algebra, the link graph is the same as the graph of ideal links, and hence is the given graph.

(d) Let $\mathfrak{g}$ be the two-dimensional solvable Lie $\mathbb{C}$-algebra with basis x, y and Lie bracket $[x, y] = x$. Then the primes of the Noetherian domain $R = U(\mathfrak{g})$ are 0, $P = xR = Rx$, and $P_\alpha = xR + (y - \alpha)R$ for any $\alpha \in \mathbb{C}$. The link graph is



Thus $\mathrm{cl}(0) = \{0\}$, $\mathrm{cl}(P) = \{P\}$, and $\mathrm{cl}(P_\alpha) = \{P_{\alpha+n} \mid n \in \mathbb{Z}\}$. One can show in each case that $\bigcap\{\, \mathcal{C}(Q) \mid Q \in \text{clique}\,\}$ is an Ore set in R.

Let's see why the link graph of $R = U(\mathfrak{g})$ looks like this. We will determine links $P_0 \rightsquigarrow P_\alpha$ for $\alpha \in \mathbb{C}$. Note $P_0 \cap P_\alpha$ equals $xR + y(y - \alpha)\mathbb{C}[y]$ unless $\alpha = 0$ (since $xR \subseteq P_0 \cap P_\alpha$ and $R/xR \cong \mathbb{C}[y]$), while P_0P_α is equal to

$$x^2R + yxR + xR(y - \alpha)R + yR(y - \alpha)R = x^2R + x(y - 1)R + x(y - \alpha)R + y(y - \alpha)\mathbb{C}[y].$$

Thus if $\alpha \neq 0, 1$, we have $P_0P_\alpha = xR + y(y - \alpha)\mathbb{C}[y] = P_0 \cap P_\alpha$, and there is no link $P_0 \rightsquigarrow P_\alpha$. For $\alpha = 1$, we have $P_0P_1 = x^2R + x(y - 1)R + y(y - 1)\mathbb{C}[y] \subset P_0 \cap P_1$. But P_0 and P_1 are maximal ideals, so this implies $P_0 \rightsquigarrow P_1$. If $\alpha = 0$, then $P_0^2 = xR + y^2\mathbb{C}[y] \subset P_0$, so again $P_0 \rightsquigarrow P_0$.

Note that we can also regard R as an Ore extension (coefficients on the left) with derivation, namely $\mathbb{C}[x][y; -x(d/dx)]$, or an Ore extension with automorphism, namely $\mathbb{C}[y][x; y \mapsto y + 1]$.

(e) The following example shows the pathology that can occur in a right Noetherian ring. Let k be a field, let $S = k[x]$, and let M be S considered as a bimodule over itself with the following actions: on the right, the action is the usual multiplication, but on the left, x acts like 0. Let R be the ring of matrices of the form $\begin{pmatrix} s & m \\ 0 & s \end{pmatrix}$ for

$s \in S, m \in M$, with the obvious addition and multiplication. The ring R is a right (but not left) Noetherian p.i. ring. Given any $p \in S$, the set of all elements of R as above such that p divides s is an ideal of R which we will denote by I_p. The prime ideals of R are precisely the ideals I_p for which $p = 0$ or p is a (monic) irreducible polynomial in S. One can show that $I_x \rightsquigarrow I_p$ for all such p and $I_p \rightsquigarrow I_p$ for all $p \neq 0$, and that these are the only links in R. Note that $I_x \rightsquigarrow I_0$ even though $I_x \supset I_0$, and that I_x is linked to infinitely many – uncountably many if k is uncountable – prime ideals. (Compare this with Conjecture 3.4 and Propositions 6.5 and 6.6.)

As we will see later, cliques in Noetherian rings are always countable. There is usually little one can say about their size beyond this. However, Sigurdsson has shown that all cliques in the differential operator ring $R[\theta; \delta]$, where R is a commutative Noetherian ring containing $\mathbb{Q}$ and δ is a derivation on R, are either singletons or countably infinite sets. Brown has shown the same result holds in universal enveloping algebras of finite dimensional solvable Lie algebras over (algebraically closed) fields of characteristic zero.

We will return to cliques later, but we first show there is another obstruction to localization, related to the one indicated by Lemma 2.5. Consider:

Example 2.8. (a) Let R be the universal enveloping algebra of the complex Lie algebra sl_2, so that R is a $\mathbb{C}$-algebra generated by elements e, f, h subject to the relations $ef - fe = h, he - eh = 2e, hf - fh = -2f$. Then R is a Noetherian domain. Let P be the augmentation ideal $eR + fR + hR$ of R, so P is a maximal ideal ($R/P \cong \mathbb{C}$) and $\mathcal{C}(P) = R \setminus P$. It is not hard to see that $h + 2 \in \mathcal{C}(P), f \in R$, and that $(h + 2)s = fd$ is not possible for $d \in \mathcal{C}(P)$, $s \in R$ (basically because $(h + 2)f = fh$ and because of the Poincaré-Birkhoff-Witt Theorem).

As noted before Lemma 2.5, the existence of links between P and other prime ideals is equivalent to the existence of certain nonsplit extensions of right ideals of the corresponding factor rings. Since $R/P \cong \mathbb{C}$, these right ideals are finite dimensional irreducible representations of sl_2 and Weyl's Theorem on complete reducibility implies such nonsplit extensions cannot exist. Thus P is not localizable and yet P is not linked to any other prime ideal (or to itself).

If we let U be the trivial representation of sl_2, then U is isomorphic to R/P as an R-module and is a homomorphic image of the Verma module $M(0)$ with trivial central character. Let M be the dual of $M(0)$ in the category $\mathcal{O}$ (the category of finitely generated right R-modules which have a basis over $\mathbb{C}$ of h-eigenvectors and for which every element is contained in a finite dimensional subspace invariant under the action of e). Then M has finite length, U is the unique simple submodule of M, and the annihilator of M is the minimal primitive ideal $Q = (P \cap \operatorname{cen} R)R$, which is strictly contained in P. Thus there is a short exact sequence $0 \to U \to M \to N \to 0$ of finitely generated uniform right R-modules with $U \subseteq R/P$, $U = \{ m \in M \mid mP = 0 \}$, $P \supset Q = \operatorname{ann}(M) = \operatorname{ann}(N)$.

(b) Let k be a field of characteristic 0 and B be the differential operator ring $k(t)[\theta; d/dt]$. Then B is a simple principal ideal domain. Set $R = \theta B + k = \mathbb{I}_B(\theta B)$. The module B/R is simple as a right R-module. One can use this fact to show that R is a Noetherian domain, whose only nontrivial ideal is θB. Note that $(\theta B)^2 = \theta B$, so by Nakayama's Lemma, θB cannot be either right or left localizable. But there are *no* links in R, so links are not the problem.

One way to look at the problem is this: let R act on $k(t)$ in the natural way from the right; then k is a submodule on which θ acts trivially, so we have an exact sequence $0 \to k \to k(t) \to L \to 0$ of right R-modules. The modules $k(t)$ and L are generated by t as R-modules (L is simple): they are uniform, faithful, torsion R-modules. The module k is a torsionfree, faithful, simple $R/\theta B$-module. This is the same situation as in (a).

In general, suppose P is a classically right localizable prime ideal in R, Q is a prime ideal of R with $Q \subset P$, and there exists a finitely generated uniform right R-module M, with $\operatorname{ann}(M) = Q$, containing a copy U of a nonzero right ideal of R/P. By passing to R/Q, we may assume $Q = 0$. We can localize at $\mathcal{C}(P)$ and get the simple R_P/PR_P-module $U \otimes R_P$ inside $M \otimes R_P$, so there is an n with $(M \otimes R_P)P^n R_P = 0$. This implies MP^n is $\mathcal{C}(P)$-torsion, so $MP^n \cap U = 0$. Thus $MP^n = 0$ and hence $P^n = 0$. This contradiction shows that we have another obstruction to localization.

We say $P \in \operatorname{Spec} R$ satisfies the *right second layer condition* (see Lemma 4.1 for an explanation of the name) if the above situation does not occur, i.e., there do not exist a prime ideal Q properly contained in P and a short exact sequence $0 \to L \to M \to N \to 0$ of finitely generated uniform right R-modules with L torsionfree as a right R/P-module, $L = \{\, m \in M \mid mP = 0 \,\}$, and $Q = \operatorname{ann}(M) = \operatorname{ann}(N')$ for every nonzero submodule N' of N. If the above situation does not occur even if we weaken the requirement that L be torsionfree to the requirement that $\operatorname{ann}(L') = P$ for every nonzero submodule L' of L, we say P satisfies the *right strong second layer condition*. (In checking the second layer condition, it is frequently convenient to note that if C is a right denominator set in R disjoint from P, then P satisfies the right second layer condition in R if and only if PC^{-1} satisfies the right second layer condition in RC^{-1}. It seems likely that this fails for the strong second layer condition.)

Theorem 2.9 (Jategaonkar)**.** *Let R be a Noetherian ring and P a prime ideal of R. Then P is classically right localizable if and only if $\{P\}$ is right link-closed and P satisfies the right second layer condition.* ∎

To prove this theorem, we need to develop some other ideas. Before doing this, I would like to state some consequences of Theorem 2.9, and introduce an important property used in studying localization.

3. The AR property

The Artin-Rees Lemma states that if I is an ideal in a commutative Noetherian ring R and M is a finitely generated R-module, then for any submodule N of M, there is a positive integer n such that $MI^n \cap N \subseteq NI$. (Recall that the I-adic topology on M is defined by declaring the subsets MI^n to be a fundamental system of neighborhoods of 0. The Artin-Rees Lemma is equivalent to the statement that the I-adic topology on N agrees with the subspace topology induced on N by the I-adic topology on M.) For an ideal I in an arbitrary ring R, let us say that I has the *right Artin-Rees (AR) property* if the Artin-Rees Lemma holds for I and any finitely generated right R-module M.

Lemma 3.1. *Suppose R is a right Noetherian ring and I is an ideal of R.*

(a) The following conditions are equivalent:

(i) *I has the right AR property.*

(ii) *For every right ideal* K *of* R*, there is a positive integer* n *with* $K \cap I^n \subseteq KI$*.*
(iii) *For any finitely generated right* R*-module* M *containing an essential submodule* L *with* $LI = 0$*, there is a positive integer* n *such that* $MI^n = 0$*.*

(b) Let $R^*(I)$ *be the subring* $R + tI + t^2I^2 + \cdots$ *of* $R[t]$*. If* $R^*(I)$ *is right Noetherian, then* I *has the right AR property and the completion of* R *with respect to its* I*-adic topology is a right Noetherian ring.*

Remark . $R^*(I)$ is called the *Rees ring of* I*.*

Proof. These results are standard. We will show only that I has the right AR property when $R^*(I)$ is right Noetherian. Let K be a right ideal of R, and for each n, set $J_n = \sum_{m=0}^{n}(K \cap I^m)t^m + \sum_{m=n+1}^{\infty}(K \cap I^n)I^{m-n}t^m$. It is not hard to see that each J_n is a right ideal of $R^*(I)$ and that $J_0 \subseteq J_1 \subseteq \cdots$. Thus by right Noetherianness of $R^*(I)$, there is an n with $J_{n-1} = J_n$. On examining the coefficients of t^n, we see that $K \cap I^n = (K \cap I^{n-1})I \subseteq KI$. ∎

If the Rees ring $R^*(I)$ of an ideal I is right Noetherian, we say I has the *right very strong AR property*. I do not know of a right AR ideal in a Noetherian ring which does not have the right very strong AR property.

If P is a prime ideal of R with the right AR property, then P satisfies the right second layer condition. For, if not, there is a finitely generated right R-module M, whose annihilator is a prime ideal strictly contained in P, which contains an essential submodule L with $LP = 0$. By the AR property, $MP^n = 0$ for some n, whence $P^n \subseteq \text{ann}(M)$. Since $\text{ann}(M)$ is prime, this implies $P \subseteq \text{ann}(M)$, contradicting our assumptions.

Links are also obstructed by AR ideals. Suppose A and B are ideals of R with $A \subset B$, P and Q are prime ideals of R with $\text{r-ann}(B/A) = P$ and $\text{l-ann}(B/A) = Q$, and suppose I is a right AR ideal of R contained in P. For some n, we have $I^nB \subseteq I^n \cap B \subseteq BI \subseteq A$, so $I^n \subseteq Q$, and hence $I \subseteq Q$. Thus if $Q \rightsquigarrow P$ and P contains an AR ideal I, then Q must also contain I. In particular, if $Q \rightsquigarrow P$ and P is right AR, then we must have $P \subseteq Q$. This leads to:

Proposition 3.2. *If* R *is a Noetherian ring and* P *is a prime ideal in* R *with the right AR property, then* P *is classically right localizable if and only if there is no prime ideal* Q *of* P *with* $P \subset Q$ *and* $Q \rightsquigarrow P$*.* ∎

Corollary 3.3. *If* R *is a Noetherian ring in which every prime ideal has the (right and left) AR property, every prime ideal of* R *is classically localizable.*

Proof. By the above, $Q \rightsquigarrow P$ implies $Q = P$ or else Q and P are incomparable, so Proposition 3.2 applies. ∎

These results are made even more interesting by the following open question:

Conjecture 3.4. *If* R *is a Noetherian ring, there do not exist prime ideals* Q *and* P *with* $Q \rightsquigarrow P$ *and either* $Q \subset P$ *or* $P \subset Q$*. More generally, there do not exist* Q *and* P *in the same clique with either* $Q \subset P$ *or* $P \subset Q$*.*

Corollary 3.5 (Another open question)**.** *If* R *is a Noetherian ring and* P *is a prime ideal of* R *with the AR property, then* P *is classically localizable.*

This last open question can be reduced to the question of whether a nilpotent prime ideal in a Noetherian ring is (classically) localizable. P. F. Smith has shown that if P has the right AR property, then $\{\, 1-p \mid p \in P \,\}$ is a right Ore set. (Conjecture 3.4 is false for right Noetherian rings – see Example 2.7(e).)

Conjecture 3.4 can be answered positively in many cases. In particular, we have the following result of Jategaonkar (see Corollary 7.3): if R is a Noetherian ring and every prime ideal of R satisfies the second layer condition, then whenever Q and P are prime ideals of R which are in the same clique, we have $\operatorname{cl.Krull.dim}(R/Q) = \operatorname{cl.Krull.dim}(R/P)$. In particular, either $Q = P$ or Q and P are incomparable.

Using Lemma 3.1 and the Hilbert Basis Theorem, McConnell showed any ideal in a Noetherian ring generated by central elements or even commuting normal elements has the very strong AR property. (An element $x \in R$ is *normal* if $xR = Rx$.) One can generalize that result as follows. An ideal I of R has a *centralizing (respectively normalizing) set of generators* if $I = x_1R + \cdots + x_nR$ where x_1 is in the center of R (respectively is normal in R) and for each k, the element x_k is central (resp. normal) modulo $x_1R + \cdots + x_{k-1}R$.

Lemma 3.6. *Let R be a right Noetherian ring and I be an ideal of R. If either I has a centralizing set of generators or I is generated by normal elements, then I has the right AR property.*

Proof. Let $I = x_1R + \cdots + x_nR$ and let M be a finitely generated right R-module with essential submodule N such that $NI = 0$.

(a) Suppose the x_i are a centralizing set of generators. We may assume by Noetherian induction that the lemma is true for all proper factors of M and over all proper factors of R. Since $Nx_1R = 0$, we have $Mx_1^t = 0$ for some t (since x_1R has the right AR property by centrality of x_1). If $Mx_1 = 0$, then M is an R/x_1R-module and we are done by induction. Suppose $Mx_1 \neq 0$ and consider the exact sequence $0 \to K \to M \to Mx_1 \to 0$ where $K = \operatorname{ann}_M(x_1) \neq 0$. Now $(Mx_1 \cap N)I = 0$ and $Mx_1 \cap N$ is an essential submodule of Mx_1. By our induction hypothesis on M, we get that $(Mx_1)I^s = 0$ for some s, so $MI^s \subseteq K$. In addition, $K \cap N$ is an essential submodule of K annihilated by I. Since K is an R/x_1R-module, we can apply our induction hypothesis to conclude that $K(I/x_1R)^{s'} = 0$ for some s'. Thus if $a = s + s'$, we have $MI^a = 0$.

(b) Suppose the x_i are all normal. If we define $\theta_i : M \to M$ by $\theta_i(m) = mx_i$, then θ_i is not a homomorphism, but $\ker \theta_i$ and $\operatorname{im} \theta_i$ are submodules of M. This also applies to powers of θ_i, so by a Fitting's Lemma type of argument, one can show $\operatorname{im} \theta_i^{k(i)} = 0$ for all sufficiently large $k(i)$. It follows that if $a = k(1) + \cdots + k(n)$, we have $MI^a = 0$. ∎

It is known that if R is a commutative Noetherian ring and G is a finitely generated nilpotent group, then every ideal in the group ring RG has a centralizing set of generators (this can be proved by a "shortest length" argument). Similarly, if k is a field, R is a commutative Noetherian k-algebra, and $\mathfrak{g}$ is a finite dimensional nilpotent Lie k-algebra, then every ideal in the enveloping algebra $R \otimes_k U(\mathfrak{g})$ has a centralizing set of generators by Engel's Theorem. Thus in both RG and $R \otimes U(\mathfrak{g})$, all prime ideals are classically localizable. Likewise all prime ideals in a principal ideal ring or Dedekind prime ring are classically localizable.

P. F. Smith has shown that the augmentation ideal of the integral group ring $\mathbb{Z}S_3$ of the symmetric group on three letters is classically localizable but fails to have the AR property. Note that this a semiprime Noetherian p.i. ring.

Before discussing the ideas needed to prove Theorem 2.9, I would like to give one amusing application of localization. Recall that in a commutative Noetherian domain, the intersection of the powers of any proper ideal is 0. (One can show that if R is a prime right Noetherian ring and I is an ideal which is contained in a proper ideal with the right AR property, then the intersection of the powers of I is 0. See Section 8 for references to some other results.)

Proposition 3.7. *If R is a prime right Noetherian ring in which every maximal ideal is classically right localizable, then $\cap_{n=1}^{\infty} I^n = 0$ for every proper ideal I of R.*

Proof. Clearly it is enough to consider the case where I is maximal. As R is prime, $\mathcal{C}(I) \subseteq \mathcal{C}(0)$, and so R embeds in the localization R_I. We know that $I^n = I^n R_I \cap R = J(R_I)^n \cap R$. Thus it's enough to show the intersection K of the powers of the Jacobson radical J of the localized ring is 0. By classicality J has the right AR property, and so for some n, we have $K = K \cap J^n \subseteq KJ$. Nakayama's Lemma now implies $K = 0$. ∎

Thus, for example, if R is a prime Noetherian ring in which every maximal ideal has the right AR property, the intersection of the powers of any proper ideal of R is 0. In particular, the intersection of the powers of any proper ideal is 0 in the enveloping algebra of a finite-dimensional nilpotent Lie algebra. By contrast, note that if $\mathfrak{g}$ is the two-dimensional solvable Lie algebra of Example 2.3 and R is its universal enveloping algebra, then the ideal $I = xR + yR$ is a maximal ideal of R and the intersection of the powers of I is xR.

4. Jategaonkar's Main Lemma

To prove Theorem 2.9 and proceed with a deeper examination of localization, we start with a technical lemma. We first introduce some terminology. If M is a uniform right module over a right Noetherian ring R, then M contains a nonzero submodule U with $\operatorname{ann}(U)$ as big as possible. It is clear that all nonzero submodules of U then have the same annihilator as U and that $\operatorname{ann}(U)$ is prime. The ideal $\operatorname{ann}(U)$ is called the *assassinator of M* (it is unique) and is denoted $\operatorname{ass}(M)$. Note $\operatorname{ass}(U) = \operatorname{ann}(U) = \operatorname{ass}(M)$. If M is an arbitrary R-module, we (abusing notation) use $\operatorname{ass}(M)$ to denote the set of assassinator primes of uniform submodules of M.

Since U is uniform, it is either torsion or torsionfree as a module over $R/\operatorname{ann}(U)$. In the first case we say M is *wild* and in the second case we say M is *tame*. It is clear that assassinators and tameness or wildness are not affected by passing to nonzero submodules or essential extensions. (In particular, we could replace M by its injective hull $E(M)$.)

Lemma 4.1 ("Main Lemma" of A. V. Jategaonkar)**.** *Let R be a right Noetherian ring, A an ideal of R, and P and Q prime ideals of R. Suppose there is an exact sequence $0 \to L \to M \to N \to 0$ of right R-modules such that (i) L is an essential submodule of M, (ii) $\operatorname{ann}(L) = P$, $\operatorname{ann}(M) = A$, $\operatorname{ann}(N) = Q$, (iii) $L = \operatorname{ann}_M(P) = \{ m \in M \mid mP = 0 \}$, (iv) L is $\mathcal{C}(P)$-torsionfree, and (v) if M' is any submodule of M, either $\operatorname{ann}(M') = A$ or $M' \subseteq L$. Then exactly one of the following conditions holds:*

(a) $Q \rightsquigarrow P$ via $A \subset Q \cap P$. If R is Noetherian or right FBN, N is $\mathcal{C}(Q)$-torsionfree.

(b) $Q = A \subset P$ *and both* M *and* N *are* $\mathcal{C}(Q)$*-torsion.*

Note that (b) cannot hold if P satisfies the right second layer condition – in fact the right second layer condition is equivalent to the condition that (b) never holds (for a fixed P). (If we think of L as the first layer of M, we may think of N as being part of the second layer of M, helping to explain the name second layer condition; these "layers" appear most clearly in the injective hull of a uniform module over a commutative Noetherian ring.) Jategaonkar has modified his definition of the right second layer condition by declaring that P satisfies it if in the Main Lemma, N is always $\mathcal{C}(Q)$-torsionfree. This new version is possibly stronger than the one used in these notes. If R is Noetherian, Jategaonkar's definition and ours are equivalent by the Main Lemma; if R is only right Noetherian, their equivalence is an open problem.

Proof. Note that $NQ = 0$, so $MQ \subseteq L$; since $LP = 0$, we see that $MQP = 0$. Clearly $A \subseteq Q$ and $A \subset P$, since $NA = LA = 0$ and $N \neq 0$. Thus $QP \subseteq A \subseteq Q \cap P$.

(b) Suppose $A = Q \cap P$. Then $PQ \subseteq A$, so $MPQ = 0$. Now $MP \neq 0$ since $A \subset P$, and either $MP \subseteq L$ and hence $\operatorname{ann}(MP) = P$, or $\operatorname{ann}(MP) = A$. Since $MPQ = 0$, we have $Q \subseteq P$ in either case. Since $A \neq P$, we must have $A = Q \subset P$. Now M contains the essential R/Q-submodule L which is torsion (since $P \cap \mathcal{C}(Q)$ is nonempty), so M is torsion as an R/Q-module.

(a) Suppose $A \subset Q \cap P$, and let $A \subset T \subseteq Q \cap P$. If T/A is right $\mathcal{C}(P)$-torsion, then clearly MT is $\mathcal{C}(P)$-torsion. This is impossible, however, since MT is a nonzero submodule of L. This proves $Q \cap P/A$ is torsionfree as a right R/P-module. (The same argument shows that Q/A is torsionfree as a right R/P-module.)

Now suppose $I(T/A) = 0$. Then $IT \subseteq A$, so $MIT = 0$. Since $T \supset A$, we must have $MI \subseteq L$, so $NI = 0$. This shows $I \subseteq Q$, so every nonzero subbimodule of $Q \cap P/A$ (or more generally of P/A) is faithful as a left R/Q-module. This shows $Q \rightsquigarrow P$.

Torsionfreeness is automatic if R is right FBN. Suppose R is Noetherian and N is not $\mathcal{C}(Q)$-torsionfree. Let $n \in N \setminus \{0\}$ and let $d \in \mathcal{C}(Q)$ with $nd = 0$. This means there is an $m \in M \setminus L$ with $md \in L$. Define a map $f : R \to M$ by $f(r) = mr$. Clearly $\ker f$ contains $d(Q \cap P) + A$, so $m(Q \cap P)$ is a factor of $Q \cap P/(d(Q \cap P) + A)$. Now $m(Q \cap P)$ is a nonzero submodule of L ($m \notin L$ so $\operatorname{ann}(mR) = A$) and hence $m(Q \cap P)$ is not $\mathcal{C}(P)$-torsion. On the other hand, the endomorphism of $Q \cap P/A$ defined by $x + A \mapsto dx + A$ is injective since $Q \cap P/A$ is left $\mathcal{C}(Q)$-torsionfree (because R is Noetherian and $Q \rightsquigarrow P$ via $A \subset Q \cap P$). Thus $(d(Q \cap P) + A)/A$ is an essential right submodule of $Q \cap P/A$ and so $Q \cap P/(d(Q \cap P) + A)$ is singular as a right R/P-module, contrary to the fact that $m(Q \cap P)$ is not $\mathcal{C}(P)$-torsion. This contradiction shows N is torsionfree over R/Q. ∎

There is a second version of the Main Lemma for Noetherian rings in which the hypothesis (iv) that L is $\mathcal{C}(P)$-torsionfree is replaced by the weaker hypothesis that $\operatorname{ass}(L) = P$. In this case we can still conclude that either (a) $Q \rightsquigarrow P$ via $A \subset Q \cap P$ or (b) $Q = A \subset P$, and case (b) cannot occur if P satisfies the *strong* right second layer condition.

We conclude from the Main Lemma that if a certain type of exact sequence exists, there is a link $Q \rightsquigarrow P$. The next result gives a converse to this.

Lemma 4.2. *Suppose* R *is a right Noetherian ring and* Q *and* P *are prime ideals of* R. *Then there is a link* $Q \rightsquigarrow P$ *if and only if there is an exact sequence* $0 \to L \to$

$M \to N \to 0$ of finitely generated uniform right R-modules with L a torsionfree right R/P-module, N a right ideal of R/Q, and $L = \operatorname{ann}_M(P)$. If R is Noetherian, we may take L to be a right ideal of R/P.

Proof. ($\Leftarrow$) Reduce M until the Main Lemma applies. ($\Rightarrow$) (This proof was suggested by K. R. Goodearl.) Let the link be via $A \subset Q \cap P$ and note that we may as well assume $A = 0$. Then Q is an essential right ideal of R since if $Q \cap I = 0$ for a right ideal I of R, then $0 = Q \cap I \supseteq I(Q \cap P)$, implying $I \subseteq Q$. Since $QP = 0$, Q is a right R/P-module, and it is not hard to see it is torsionfree.

Set $E = E(R_R)$ and $K = \operatorname{ann}_E(P)$. It is easy to see that $K \cap R = \text{l-ann}_R(P) = Q$ and so $R/Q = R/K \cap R \cong R + K/K$ embeds in E/K. We can write $E = E_1 \oplus \cdots \oplus E_n$ for some uniform injectives $E_1, \ldots, E_n$, and clearly we have $K = (K \cap E_1) \oplus \cdots \oplus (K \cap E_n)$ by definition of K, so $E/K \cong (E_1/K \cap E_1) \oplus \cdots \oplus (E_n/K \cap E_n)$. It follows that some $E_j/K \cap E_j$ contains a copy of a uniform right ideal U of R/Q. Further, K is an essential submodule of E which is torsionfree as a right R/P-module, since Q is an essential submodule of E.

Now let M' be a finitely generated submodule of E_j such that $K \cap E_j \subseteq M' \subseteq E_j$ and $M'/K \cap E_j \cong U$ and let M be a finitely generated submodule of M' which is not contained in K. If we set $L = K \cap M = \operatorname{ann}_M(P)$ and $N = M/L \neq 0$, we get the desired exact sequence since N embeds in $M'/K \cap E_j$. ∎

Thus if $Q \rightsquigarrow P$, we have $\operatorname{Ext}^1(V, U) \neq 0$ for some uniform right ideals U of R/P and V of R/Q. In general the converse need not hold, since only extensions of a special kind are relevant. However, if $\operatorname{cl.Krull.dim}(R/Q) = \operatorname{cl.Krull.dim}(R/P)$ and every prime ideal in R satisfies the second layer condition, then the converse is valid.

The following is a very useful consequence of the Main Lemma, giving a kind of critical series for a module when it applies.

Lemma 4.3. *Let M be a finitely generated right R-module and let $X = \bigcup\{ \operatorname{rt.cl}(P) \mid P \in \operatorname{ass}(M) \}$. If either*
(i) every prime ideal in X satisfies the strong right second layer condition or
(ii) R is Noetherian, every prime ideal in X satisfies the right second layer condition, and $\operatorname{ann}_M(P)$ is a torsionfree right R/P-module for every $P \in \operatorname{ass}(M)$ (i.e., M is tame),
then M has a chain of submodules $0 = M_0 < M_1 < \cdots < M_n = M$ such that each M_i/M_{i-1} is uniform and $\operatorname{ann}(M_i/M_{i-1}) = P_i$ for some $P_i \in X$ (whence $MP_n \cdots P_1 = 0$). Moreover, in case (ii) each M_i/M_{i-1} embeds in R/P_i.

Proof. We prove case (ii). By Noetherian induction we may assume the lemma is true for any proper factor of M. Choose $P \in \operatorname{ass}(M)$ and let $L = \operatorname{ann}_M(P) \neq 0$. Let U be a maximal uniform submodule of L. It is enough to show that the $\operatorname{ass}(M/U) \subseteq X$ and that if $Q \in \operatorname{ass}(M/U)$, then $\operatorname{ann}_{M/U}(Q)$ is torsionfree as a right R/Q-module.

Let V be a uniform submodule of M/U with $\operatorname{ann}(V) = \operatorname{ass}(V) = Q$, say $V = M'/U$ for some submodule M' of M containing U. If M' is uniform, then we may shrink it if necessary and apply the Main Lemma to get that $Q \rightsquigarrow P$ and that some submodule of V, and hence V itself, is torsionfree over R/Q. If M' is not uniform, then some uniform submodule of M embeds in V, and so $Q \in \operatorname{ass}(M)$ and V is torsionfree over R/Q by the hypotheses in (ii). ∎

Warfield has noted that this enables us to prove a converse to the result mentioned before Proposition 3.2.

Proposition 4.4. *If R is a right Noetherian ring and I is an ideal in R such that all prime ideals of R containing I satisfy the strong right second layer condition, then I has the right AR property if and only if for any prime ideals Q, P of R with $Q \rightsquigarrow P$ and $P \supseteq I$, we have $Q \supseteq I$.*

Proof. ($\Rightarrow$) See the remarks before Proposition 3.2. ($\Leftarrow$) Suppose M is a finitely generated right R-module which contains an essential submodule L with $LI = 0$. Clearly every assassinator prime of M contains I, and so the last lemma implies $MI^n = 0$ for some n. ∎

Corollary 4.5. *If R is a right Noetherian ring and P is a prime ideal of R for which R/P is Artinian, then P has the right AR property if and only if P is classically right localizable.* ∎

Can we change the hypothesis that R/P is Artinian in the last corollary to the hypothesis that P is a maximal ideal ?

There are many rings in which it is possible to show that every prime ideal satisfies the second layer condition, and hence only the "good" case of the Main Lemma can occur. We first state a general result which follows easily from the comments before Proposition 3.2 and then we state a result listing some of the rings in which the second layer condition holds. We sketch the proof in two cases here and in more cases in the appendices, including outlines of the proofs for enveloping algebras and group rings. (We remark that a proof similar to that of Proposition 4.6 shows that any prime ideal with a normalizing set of generators in a right Noetherian ring satisfies the right second layer condition.) We define a ring R to be *right AR-separated* if for any prime ideals Q, P of R with $P \supset Q$, there is an ideal I with $P \supseteq I \supset Q$ such that I/Q has the right AR property in R/Q.

Proposition 4.6. *If R is a right Noetherian right AR-separated ring, then every prime ideal of R satisfies the strong right second layer condition.* ∎

Theorem 4.7. *In each of the following types of Noetherian rings, every prime ideal satisfies the second layer condition: FBN rings (e.g., Noetherian p.i. rings); Artinian rings; principal ideal rings; Hereditary Noetherian Prime rings with enough invertible ideals; group rings RG or enveloping algebras $R \otimes U(\mathfrak{g})$, where R is commutative Noetherian, G is a polycyclic-by-finite group, and $\mathfrak{g}$ is a finite dimensional (solvable in characteristic 0) Lie algebra (or superalgebra); Ore extensions $R[\theta; \delta]$, $R[x; \phi]$, $R[x, x^{-1}; \phi]$, where R is a commutative Noetherian ring, δ is a derivation on R, and ϕ is an automorphism of R; and the group-graded and skew enveloping analogues of the above rings. Conversely, if $\mathfrak{g}$ is a nonsolvable Lie algebra over a field of characteristic 0 and $R = U(\mathfrak{g})$ or R is a Hereditary Noetherian Prime ring without enough invertible ideals, then there are prime ideals in R which do not satisfy the right or left second layer condition.*

Partial proof. (i) Let R be an FBN ring, let Q, P be prime ideals of R with $Q \subset P$, and let M be a finitely generated uniform right R-module with $\operatorname{ann}(M) = Q$ such that M contains a submodule U with $\operatorname{ann}(U) = P$. Since U is torsion as an R/Q-module, M is torsion as an R/Q-module. The FBN property now implies that each of the finitely many generators of M is annihilated by an ideal strictly bigger than Q, and so $\operatorname{ann}(M) \supset Q$. This contradiction shows that P satisfies the second layer condition.

(ii) Let $T = R[\theta;\delta]$, assuming R is a commutative Noetherian ring and $R \supseteq \mathbb{Q}$. Suppose Q and P are prime ideals of T with $Q \subset P$. We know $Q \cap R$ is a δ-invariant prime ideal of R (since $R \supseteq \mathbb{Q}$), so we may pass to $R/Q \cap R$ and hence we may assume $Q \cap R = 0$ and R is a commutative Noetherian domain. Suppose $Q \neq 0$: then let C be the set of nonzero elements of R. It's not hard to show that C is an Ore set of regular elements in T. If we localize, we have $TC^{-1} = RC^{-1}[\theta;\delta]$ and now RC^{-1} is a field. Also QC^{-1} is a nonzero proper ideal of TC^{-1}, so TC^{-1} is not simple. This implies $\delta = 0$.

Thus if $Q \neq 0$ and $Q \cap R = 0$, we have $\delta = 0$ and so T is commutative. In this case all prime ideals in T satisfy the second layer condition. Thus we may assume $Q = 0$. Now if $P \cap R = 0$, we know $P \neq 0$ and so T is commutative again. Thus if the second layer condition is to fail, we must have a uniform finitely generated faithful T-module M containing a copy of a nonzero right ideal U of T/P where $P \cap R = I \neq 0$. We will show IT has the AR property. Presuming this is so, we have $U(IT) = 0$, which implies $M(IT)^n = 0$ for some n. Since M is faithful, this means $I^n = 0$, which is impossible.

All we have to do to finish is show IT is AR. To show this, it suffices to show the ring $T^*(IT) = T + xIT + x^2I^2T + \cdots$ is Noetherian. It is easy to see that $T^*(IT) = (R^*(I))[\theta;\delta]$, where $R^*(I) = R + xI + x^2I^2 + \cdots$ and we extend δ to $R^*(I)$ by defining $\delta(x) = 0$. By the Hilbert Basis Theorem, $R^*(I)$ and $(R^*(I))[\theta;\delta]$ are Noetherian. This shows IT is AR.

(iii) We saw in Example 2.8(a) that the augmentation ideal of $U(sl_2)$ does not satisfy the second layer condition. Since any nonsolvable Lie algebra has a simple Lie algebra as a factor, one can use the same ideas to show that the augmentation ideal in the universal enveloping algebra of a nonsolvable complex Lie algebra never satisfies the second layer condition. ∎

Note that we have in fact proved that if Q, P are prime ideals in $T = R[\theta;\delta]$ and P is minimal over Q, then P/Q is an AR ideal in T/Q. (If R is a Noetherian p.i. ring and if Q and P are prime ideals in R with $Q \subset P$, then one can use Rowen's Theorem to show there is an ideal I with $Q \subset I \subseteq P$ such that I/Q is an AR ideal in R/Q.) Thus every prime ideal in T actually satisfies the strong second layer condition. The strong second layer condition is known to hold in the types of rings listed in the theorem, except in the general group-graded and skew enveloping cases.

We also have the following important general result of E. S. Letzter (see Theorem 7.8): suppose R is a Noetherian subring of a ring S such that S is finitely generated as both a right and left R-module. If every prime ideal of R satisfies the [strong] second layer condition, then so does every prime ideal of S.

We know $T = R[\theta;\delta]$ is nice, since all primes satisfy the second layer condition, but it can have large cliques. For example, if $\mathfrak{g}$ is the complex two-dimensional solvable Lie algebra of Example 2.3, then in $T = \mathbb{C}[x][\theta; x(d/dx)] = U(\mathfrak{g})$, almost all link components are infinite – only two cliques are singletons and hence very few prime ideals are localizable. We need a more general notion of localizing at a prime ideal.

5. Localizing at cliques

Given any prime ideal P, any right localization at P must be gotten by inverting a right Ore set C contained in $\mathcal{C}(P)$. Thus in fact $C \subseteq \bigcap\{\, \mathcal{C}(Q) \mid Q \in \operatorname{rt\,cl}(P) \,\}$ by a previous result. If $X \subseteq \operatorname{Spec} R$, define $\mathcal{C}(X) = \bigcap\{\, \mathcal{C}(Q) \mid Q \in X \,\}$. Thinking of X as

a clique, if we want to localize at it, we might ask that $\mathcal{C}(X)$ be right Ore. We also want some nice properties for the quotient ring, so we make the following definition.

If $X \subseteq \operatorname{Spec} R$, we say X is *right localizable* if there is a right Ore set C disjoint from every $P \in X$ such that the localization RC^{-1} has the following properties:
(a) For every $P \in X$, the ring RC^{-1}/PRC^{-1} is Artinian.
(b) The only right primitive ideals are the ideals PRC^{-1} for $P \in X$.
If in addition RC^{-1} satisfies:
(c) Every finitely generated right RC^{-1}-module which is an essential extension of a simple right RC^{-1}-module is Artinian,
then we say X is *right classically localizable*. (We remark in advance that the proofs below show that if X is right localizable, then X is classically right localizable if and only if every prime ideal in X satisfies the right second layer condition.)

Note that (a) is equivalent to the condition that RC^{-1}/PRC^{-1} is the Goldie quotient ring of R/P and that (c) is equivalent to the condition that the injective hull of a simple right RC^{-1}-module is the union of its socle sequence. Any such C must be contained in $\mathcal{C}(X)$, and it is not hard to see that if X is right localizable, then $\mathcal{C}(X)$ is a right denominator set and $RC^{-1} = R\mathcal{C}(X)^{-1}$. We denote this localization by R_X. For a finite set X of incomparable primes, localizing at the set X is the same thing as localizing at the semiprime ideal $\cap X$.

Before proceeding let us note one interesting fact.

Proposition 5.1. *If R is a right FBN ring and C is a right denominator set in R which does not contain 0, then there is a classically right localizable set X of prime ideals of R such that $C \subseteq \mathcal{C}(X)$ and $RC^{-1} = R_X$.*

Proof. Let X be the set of ideals of R which are maximal with respect to being disjoint from C. It is not hard to see that the ideals of X are prime ideals, and that the set of maximal ideals of RC^{-1} is $\{ PC^{-1} \mid P \in X \}$. Since C is disjoint from every element of X, we have $C \subseteq \mathcal{C}(X)$. By Lemma 2.1, RC^{-1} is right FBN, and so a prime ideal Q of RC^{-1} is primitive or maximal if and only if RC^{-1}/Q is Artinian. In addition, any finitely generated essential extension of a simple right RC^{-1}-module is Artinian by the FBN property, so RC^{-1} is a classical right localization of X. ∎

We will see that in many Noetherian rings all cliques are classically localizable. It may be that in any Noetherian ring, every clique in which all prime ideals satisfy the second layer condition is classically localizable: that is an open question at this time. To clarify the situation, we need to introduce another condition.

If $X \subseteq \operatorname{Spec} R$, we say X satisfies the *right second layer condition* if every prime ideal in X satisfies it, and we say X satisfies the *incomparability condition* if there do not exist prime ideals $Q, P \in X$ with $Q \subset P$. We say X satisfies the *right intersection condition* if for any right ideal I of R such that $I \cap \mathcal{C}(P)$ is nonempty for every $P \in X$, the intersection $I \cap \mathcal{C}(X)$ is nonempty. Following ideas of Müller, we say X satisfies the *right weak intersection condition* if for any right ideal I of R with the property that for every $P \in X$ and every $r \in R$ the intersection $I \cap r\mathcal{C}(P)$ is nonempty, we have $I \cap \mathcal{C}(X)$ nonempty. This condition may be stated in another way which makes clear its adequacy for our proofs: any right R-module which is $\mathcal{C}(P)$-torsion for all $P \in X$ is $\mathcal{C}(X)$-torsion.

Because of his re-definition of the second layer condition (see comments after the Main Lemma), Jategaonkar states the following proposition and the theorem after it for right Noetherian rings. With our definition of the second layer condition, however, the proof uses part of the Main Lemma where the ring is required to be Noetherian (see the third paragraph of the proof). I do not know whether this result is valid for right Noetherian rings; it is valid for those in which the torsionfreeness in the Main Lemma is automatic, for example for right Noetherian p.i. rings.

Proposition 5.2. *If R is a Noetherian ring and X is a right link-closed subset of* $\operatorname{Spec} R$ *satisfying the weak right intersection condition and the right second layer condition, then $\mathcal{C}(X)$ is a right denominator set in R.*

Proof. Let $C = \mathcal{C}(X)$ and suppose it is not right Ore. Then there is a $c \in C$ such that the module R/cR is not C-torsion. Using the Noetherianness of R/cR, we can find a factor K of R/cR such that K is not C-torsion but any proper factor of K is C-torsion. This module K is uniform and cyclic with a generator which is C-torsion. By the weak intersection condition, there is a $P \in X$ for which K is not $\mathcal{C}(P)$-torsion.

Set $L = \{\, k \in K \mid kP = 0 \,\}$. We will show $L = K$. Suppose $L \subset K$. Since L is either zero or a uniform R/P-module, it is either $\mathcal{C}(P)$-torsionfree or $\mathcal{C}(P)$-torsion. Suppose $x \in K$ is not $\mathcal{C}(P)$-torsion. Then $(\operatorname{ann}(x)+P)/P$ is not an essential right ideal of R/P, so $R/(\operatorname{ann}(x)+P)$ is not a torsion R/P-module. Since $R/(\operatorname{ann}(x)+P) \cong xR/xP$ and K/xP contains xR/xP, this would imply K/xP is not $\mathcal{C}(P)$-torsion. This is impossible unless $xP = 0$, so $x \in L$. Thus we see that $L \neq 0$ and that L is $\mathcal{C}(P)$-torsionfree.

Still under the assumption that $L \subset K$, reduce K to a submodule M containing L such that the Main Lemma applies to the exact sequence $0 \to L \to M \to M/L \to 0$, and let $Q = \operatorname{ann}(M/L) = \operatorname{ass}(M/L)$. By the Main Lemma and the fact that P satisfies the right second layer condition, we see that $Q \rightsquigarrow P$ via $\operatorname{ann}(M) \subset Q \cap P$ and that M/L is $\mathcal{C}(Q)$-torsionfree (this is where we use two-sided Noetherianness). Since X is right link-closed, we see that $Q \in X$ and so M/L is $\mathcal{C}(X)$-torsionfree. Thus K/L is not $\mathcal{C}(X)$-torsion, contradicting the fact that $L \neq 0$.

This shows that $L = K$, i.e., $KP = 0$. But this makes K an R/P-module which is not torsion but is generated by a torsion element. This contradiction shows that C must be right Ore after all. ∎

Here is the main result. Again, it is valid for right FBN rings and may hold for general right Noetherian rings. Note that it has Theorem 2.9 as a corollary.

Theorem 5.3 (Jategaonkar)**.** *If R is a Noetherian ring and $X \subseteq \operatorname{Spec} R$, then X is classically right localizable if and only if (i) X is right link-closed, (ii) X satisfies the right second layer condition, (iii) X satisfies the (weak) right intersection condition, and (iv) X satisfies incomparability.*

Proof. ($\Leftarrow$) We've shown that $R_X = RC^{-1}$ exists for $C = \mathcal{C}(X)$. Let I be a right ideal of R containing P with I/P essential in R/P. Then $I \cap r\mathcal{C}(P)$ is nonempty for every $r \in R$, since $r + I$ is in the torsion module R/I. Suppose $Q \in X$ and $Q \neq P$. By incomparability, $(P+Q)/Q$ is a nonzero ideal of R/Q, and so is essential as a right ideal. Thus $P \cap r\mathcal{C}(Q)$ is nonempty for every $r \in R$, and hence so is $I \cap r\mathcal{C}(Q)$. The weak intersection condition now implies that $I \cap \mathcal{C}(X)$ is nonempty. This shows that

every essential right ideal of $\bar{R} = R/P$ meets $\bar{C} = \{ c + P \mid c \in \mathcal{C}(X) \}$, so $\bar{R}\bar{C}^{-1}$ is the Goldie quotient ring of $\bar{R}$. Thus R_X/PR_X is Artinian, as it is isomorphic to $\bar{R}\bar{C}^{-1}$.

Suppose N is a simple right R_X-module. Then N is certainly a uniform C-torsionfree right R-module, and furthermore, any proper R-module factor of N is C-torsion. Thus as in the proof of the last proposition, we have $NP = 0$ for some $P \in X$. This implies $\operatorname{ann}_R(N) = P$, since N is C-torsionfree, and hence the R_X-annihilator of N is PR_X.

Suppose now that M is a finitely generated essential R_X-module extension of N. Then there are finitely generated uniform R-submodules N', M' of N, M such that $N' \subseteq M'$ and $N'R_X = N$, $M'R_X = M$. Clearly $\operatorname{ass}(M') = \operatorname{ass}(N') = P$ and N' is torsionfree as a right R/P-module, since NR_X embeds in R_X/PR_X. Thus by Lemma 4.3, there are prime ideals $P_1, \ldots, P_n \in X$ with $M'P_n \cdots P_1 = 0$. It follows that $MP_nR_X \cdots P_1R_X = 0$, and so M is Artinian. This proves R_X is a classical localization.

($\Rightarrow$) Incomparability holds since each R_X/QR_X is Artinian and $Q \subset P$ implies $QR_X \subset PR_X$.

Suppose $Q \rightsquigarrow P$ in R with $Q \in \operatorname{Spec} R$ and $P \in X$. Since $C = \mathcal{C}(X) \subseteq \mathcal{C}(P)$, we have $C \subseteq \mathcal{C}(Q)$. By Lemma 4.2, there is an exact sequence $0 \to L \to M \to N \to 0$ of finitely generated uniform right R-modules with $L \subseteq R/P$ and $N \subseteq R/Q$. If we localize this exact sequence with respect to C, we have an exact sequence of finitely generated uniform RC^{-1}-modules, and LC^{-1} embeds in RC^{-1}/PC^{-1}, so it is simple. Since R_X is a classical localization, MC^{-1} is Artinian, whence NC^{-1} is Artinian. Since NC^{-1} embeds in RC^{-1}/QC^{-1}, we see that QC^{-1} is primitive. Thus $Q \in X$, and so X is right link-closed. (If we know R_X is Noetherian, a shorter proof is available. We have $QR_X \rightsquigarrow PR_X$ in R_X. Since R_X/PR_X is Artinian, the result of Lenagan stated in Section 2 implies R_X/QR_X is Artinian, so certainly QR_X is primitive. Thus by classicality, $Q \in X$.)

Suppose $P \in X$ does not satisfy the right second layer condition. Then there is an exact sequence $0 \to L \to M \to N \to 0$ of finitely generated uniform right R-modules such that L is $\mathcal{C}(P)$-torsionfree and $\operatorname{ann}(M) = Q \subset P$, and so M is $\mathcal{C}(Q)$-torsion. Localizing this sequence at $C = \mathcal{C}(X)$, we get an exact sequence $0 \to LC^{-1} \to MC^{-1} \to NC^{-1} \to 0$ of finitely generated right R_X-modules. Since L is C-torsionfree and essential in M, we see that M is C-torsionfree and that MC^{-1} is uniform as an R_X-module. This shows LC^{-1} is a simple R_X/PR_X-module. By classicality, MC^{-1} is Artinian, so MC^{-1} is annihilated by $P_n \cdots P_1R_X$ for some $P_1, \ldots, P_n \in X$. Thus $MP_n \cdots P_1 = 0$, so $P_n \cdots P_1 \subseteq Q$. Hence $P_i \subseteq Q \subset P$ for some i, violating incomparability.

Finally, suppose I is a right ideal of R such that I is disjoint from $C = \mathcal{C}(X)$ but $I \cap \mathcal{C}(P)$ is nonempty for each $P \in X$. We may take I maximal with respect to this property, in which case $M = R/I$ is not C-torsion, but every proper factor of it is, and so M is C-torsionfree. This implies MC^{-1} is simple over RC^{-1}, so $\operatorname{ann}(MC^{-1}) = PR_X$ for some $P \in X$. This in turn implies M is an R/P-module generated by a $\mathcal{C}(P)$-torsion element, so M is $\mathcal{C}(P)$-torsion. On the other hand, MC^{-1} embeds in R_X/PR_X, which contradicts M being $\mathcal{C}(P)$-torsion. This contradiction shows the intersection condition holds. ∎

Theorem 5.4. *If R is a Noetherian ring and X is a clique in R, then X is classically localizable if and only if X satisfies the (weak) intersection condition and the second layer condition.*

Proof. We need to show a clique X satisfying the weak intersection condition and the second layer condition also satisfies incomparability. (Jategaonkar has in fact shown that any clique in a Noetherian ring satisfying the second layer condition satisfies incomparability.) By Proposition 5.2, we know that $C = \mathcal{C}(X)$ is an Ore set. Localize at C: we still have a clique in R_X (see remarks after the definition of links), so we may replace R by R_X. Let P be maximal in X and suppose there is a $Q \in X$ with $Q \subset P$. Then R/P is Artinian (we did not need incomparability to show that R_X/PR_X is Artinian in the proof of Theorem 5.3 ($\Leftarrow$), we only needed P to be maximal in X), and by repeatedly applying the result of Lenagan stated in Section 2, we see that R/Q is also Artinian. (Since X is a clique, we can get from P to Q via a finite series of links.) This cannot possibly be true since R/Q is not simple, so no such Q can exist. ∎

In general, this leaves us with the problem of deciding when cliques satisfy the second layer condition and the intersection condition. We can immediately give an answer in one case; the next result shows every prime ideal is classically localizable in principal ideal rings, Dedekind prime rings, and group rings RG and enveloping algebras $R \otimes U(\mathfrak{g})$ with R commutative Noetherian, G a finitely generated nilpotent group, and $\mathfrak{g}$ a finite dimensional nilpotent Lie algebra.

Corollary 5.5. *Let R be a Noetherian ring. If every prime ideal in R satisfies the second layer condition, then any prime ideal in R with the AR property is classically localizable. Thus if every prime ideal in R has the AR property, every prime ideal in R is classically localizable.* ∎

We know from Section 4 that all prime ideals in FBN rings satisfy the second layer condition. The next result shows one only needs to check a simpler version of the intersection condition in such rings.

Proposition 5.6. *Let R be a right FBN ring. A right link-closed set X of prime ideals of R is classically right localizable if and only if X satisfies incomparability and for every ideal I of R for which $I \cap \mathcal{C}(P)$ is nonempty for all $P \in X$, we have $I \cap \mathcal{C}(X)$ nonempty.*

Proof. We need only show that X satisfies the weak right intersection condition. Let M be a finitely generated (it's enough to handle this case) right R-module which is $\mathcal{C}(P)$-torsion for all $P \in X$. Set $I = \operatorname{ann}(M)$ and note I is an ideal. Since R is right FBN, we have that $I = \text{r-ann}(R/J) = \operatorname{ann}(m_1, \ldots, m_n)$ for some $m_1, \ldots, m_n \in M$. It follows that R/I embeds in M^n as a right module, which implies R/I is $\mathcal{C}(P)$-torsion for all $P \in X$. This implies $I \cap \mathcal{C}(P)$ is nonempty for all $P \in X$. Hence by our hypothesis, $I \cap \mathcal{C}(X)$ is nonempty, or equivalently, R/I is $\mathcal{C}(X)$-torsion. Since M is an R/I-module, it is a quotient of a direct sum of copies of R/I, and hence is $\mathcal{C}(X)$-torsion. ∎

Corollary 5.7. *If R is an FBN ring and X is a clique in R, then X is classically right localizable if and only if X is classically left localizable.* ∎

Müller has shown that all cliques are classically localizable in an affine Noetherian p.i. ring. The next result shows some symmetry exists in a prime Noetherian p.i. ring. We omit the proof, but remark that it is proved by showing two primes Q, P in a prime Noetherian p.i. ring R are in the same right (or left) clique if and only if there are prime

ideals Q', P' in the trace ring T of R such that $Q' \cap \operatorname{cen} T = P' \cap \operatorname{cen} T$, $Q' \cap R = Q$, and $P' \cap R = P$.

Theorem 5.8 (Braun-Warfield)**.** *If R is a prime Noetherian p.i. ring, then a set X of prime ideals of R is right link-closed if and only if it is left link-closed. Thus X is classically right localizable if and only if it is classically left localizable.* ∎

Corollary 5.9. *If R is a prime Noetherian p.i. ring and C is a right Ore set in R, then there is a right and left Ore set $D \supseteq C$ with $RC^{-1} = RD^{-1}$.*

Proof. Combine Theorem 5.8 with Proposition 5.1. ∎

Question: Can we take D to be central in Corollary 5.9? The answer is yes when R equals its trace ring.

6. The intersection condition and generic regularity

We have discussed some Noetherian rings in which all prime ideals satisfy the second layer condition. We now discuss when subsets X of R satisfy the intersection condition.

If $X = \{ P_1, \ldots, P_n \}$ is an incomparable subset of $\operatorname{Spec} R$, it is easy to see that X satisfies the intersection condition as follows. Let I be a right ideal of R with $I \cap \mathcal{C}(P_k)$ nonempty for each $k = 1, \ldots, n$, so that $I + P_k/P_k$ is an essential right ideal of R/P_k. It is not hard to see that $I_k + P_k/P_k$ is an essential right ideal of R/P_k for $I_k = I \cap \bigcap \{ P_j \mid j \neq k \}$, so there is a $c_k \in I_k \cap \mathcal{C}(P_k)$. Since $c_k \in P_m$ for $m \neq k$, it is clear that $c = \sum c_k \in I \cap \mathcal{C}(P_m)$ for each $m = 1, \ldots, n$. Thus $c \in I \cap \mathcal{C}(X)$. With a little more work one can show any finite subset of $\operatorname{Spec} R$ satisfies the intersection condition (this follows from Lemma 1.11). This leads to the following result.

Proposition 6.1. *If R is a Noetherian ring and X is a finite clique satisfying the second layer condition, then X is classically localizable.* ∎

Now we turn to the study of the intersection condition for infinite sets. The most convenient way of discussing current results is to introduce the concept of generic regularity; this in turn requires us to topologize $\operatorname{Spec} R$. Throughout this section, all topological and continuity statements refer to the patch topology on $\operatorname{Spec} R$ introduced in the next paragraph and the usual topology on the rational numbers $\mathbb{Q}$.

For a right Noetherian ring R, we define the *patch* (or *constructible*) topology on $\operatorname{Spec} R$ to be the topology for which a fundamental system of neighborhoods of each $P \in \operatorname{Spec} R$ is

$$\{ V(P) \cap W(I) \mid I \text{ is an ideal of } R \text{ properly containing } P \}$$

where $V(P) \cap W(I) = \{ Q \in \operatorname{Spec} R \mid P \subseteq Q \text{ and } I \not\subseteq Q \}$. This topology makes $\operatorname{Spec} R$ a totally disconnected compact Hausdorff space. It is not hard to show that the closure of a subset X of $\operatorname{Spec} R$ is the set of prime ideals of R which are intersections of ideals in X.

We say a subset X of $\operatorname{Spec} R$ satisfies the *generic regularity condition* if for any $T \in \operatorname{Spec} R$ and any $c \in \mathcal{C}(T)$, there is an open neighborhood $\mathcal{U}$ of T such that $c \in \mathcal{C}(P)$ for any $P \in \mathcal{U} \cap X$. (Jategaonkar calls such a set X *sparse*.) Phrased another way, X satisfies the generic regularity condition if and only if for any $T \in \operatorname{Spec} R$ and any $c \in \mathcal{C}(T)$, the intersection of all the $P \in X$ for which $P \supseteq T$ and $c \notin \mathcal{C}(P)$ properly contains T. Note that if X satisfies the generic regularity condition, then

$\{ P \in X \mid r \in \mathcal{C}(P) \}$ is an open subset of X in the subspace topology, for any $r \in R$. One can show that $\operatorname{Spec} A_1(\mathbb{Z})$ does not satisfy the generic regularity condition.

It is easy to see that a finite union of sets satisfying the generic regularity condition satisfies the generic regularity condition and hence that any finite subset of $\operatorname{Spec} R$ satisfies the generic regularity condition. In order to prove further results, we need to introduce some new ideas.

Suppose R is a right Noetherian ring, A is a finitely generated right R-module, and P is a prime ideal in R. Let Q_P denote the Goldie quotient ring of R/P. Then $A \otimes_R Q_P$ is a finitely generated right Q_P-module, so it is a Q_P-module of finite length. We define the *normalized rank of A at P* to be $\hat{\rho}(A, P) = \operatorname{length}(A \otimes_R Q_P)/\operatorname{length}(Q_P)$. Note that the length of Q_P is just the uniform dimension of R/P and the length of $A \otimes_R Q_P$ is the torsionfree rank of the R/P-module A/AP (that is, the uniform dimension of $(A/AP)/T$ where T is the torsion submodule of A/AP as an R/P-module). Clearly $\hat{\rho}(A, P) = 0$ if and only if A/AP is torsion as an R/P-module. In particular, $\hat{\rho}(R/xR, P) = 0$ if and only if $x \in \mathcal{C}(P)$.

To illustrate this concept, suppose P is a localizable prime ideal of R such that R/P is a domain (for example, suppose R is commutative) and let A_P be the localization of A at P. Then Q_P is a division ring and $\operatorname{length}(A \otimes_R Q_P)$ is the dimension of $(A/AP) \otimes_{R/P} QP$ as a Q_P-vector-space, that is, $\operatorname{length}(A \otimes_R Q_P) = \operatorname{rank}(A_P)$, so $\hat{\rho}(A, P)$ is the local rank of A at P.

Here is the main result about normalized ranks (due to Stafford). For a proof, see the paper of Goodearl mentioned in Section 8.

Theorem (Continuity Theorem)**.** *Let R be a right Noetherian ring and let A be a finitely generated right R-module. Define $\sigma_A : \operatorname{Spec} R \to \mathbb{Q}$ by $\sigma_A(P) = \hat{\rho}(A, P)$.*

(a) σ_A is continuous.

(b) If $X \subseteq \operatorname{Spec} R$ satisfies the generic regularity condition, then for any $T \in \operatorname{Spec} R$, there is a neighborhood $\mathcal{U}$ of T such that $\sigma_A(P) = \sigma_A(T)$ for all $P \in X \cap \mathcal{U}$.

(c) If $X \subseteq \operatorname{Spec} R$ and for every cyclic right R-module A, the map σ_A satisfies the conclusion of (b), then X satisfies the generic regularity condition. ∎

The theorem implies that $\operatorname{Spec} R$ satisfies the generic regularity condition if and only if σ_A is locally constant for every finitely generated right R-module A.

Proposition 6.2. *Suppose R is a right Noetherian ring and X is a subset of $\operatorname{Spec} R$. Then X satisfies the generic regularity condition if any of the following hold.*
(a) R is right FBN,
(b) $\operatorname{rt.Krull.dim}(R) \leq 1$
(c) $\sup\{ \operatorname{uniform.dim}(R/P) \mid P \in X \} < \infty$.

Proof. Suppose $T \in \operatorname{Spec} R$ and $c \in \mathcal{C}(T)$. We need to find an ideal $I \supset T$ such that whenever $P \in X$ and $P \supseteq T$ and $P \not\supseteq I$, we have $c \in \mathcal{C}(P)$. Clearly we may pass to R/T and so assume $T = 0$. It is then clear that the right ideal cR is essential.

(a) Suppose that R is right fully bounded. Then there is a nonzero ideal $I \subseteq cR$. Clearly if $P \not\supseteq I$, then $(I+P)/P$ is an essential right ideal of R/P. Thus $(c+P)(R/P)$ is an essential right ideal of R/P and so $c \in \mathcal{C}(P)$.

(b) Suppose R has right Krull dimension at most 1. If $T = 0$ is maximal, then clearly $I = R$ works. Otherwise, we see that R/cR is a module of finite length. Let

I be the intersection of the annihilators of the unfaithful simple composition factors of R/cR. (Set $I = R$ if all the simple composition factors are faithful.) Then I is a nonzero ideal of R.

Suppose $P \not\supseteq I$ and $P \neq 0$. Then P cannot annihilate any composition factor S of R/cR. It follows that $SP = S$ for each such S, and hence $\hat{\rho}(S, P) = 0$. This implies $\hat{\rho}(R/cR, P) = 0$ (since normalized rank is subadditive), and so $c \in \mathcal{C}(P)$.

(c) Set $n = \sup\{ \text{uniform dim}(R/P) \mid P \in X \}$. There is a neighborhood $\mathcal{U}$ of T such that if $A = R/cR$, then $\sigma_A(P) = |\sigma_A(T) - \sigma_A(P)| < 1/n$ whenever $P \in \mathcal{U}$. But if $P \in X$, then $\sigma_A(P)$ is a nonnegative rational number whose denominator is at most n, so this inequality implies $\sigma_A(P) = 0$. Thus $c \in \mathcal{C}(P)$ whenever $P \in \mathcal{U} \cap X$. ∎

As noted in Appendix A, if R is a commutative Noetherian algebra over a field k of characteristic zero and $\mathfrak{g}$ is a finite dimensional solvable Lie k-algebra, then for every prime ideal P in $T = R \otimes U(\mathfrak{g})$ or more generally $T = R\#_\sigma U(\mathfrak{g})$, the ring T/P is a domain and hence has uniform dimension 1. Thus any subset of $\operatorname{Spec} T$ satisfies the generic regularity condition.

There are two reasons for introducing the generic regularity condition. One is that subsets of $\operatorname{Spec} R$ satisfying it frequently satisfy the intersection condition. The other is that using generic regularity, one can prove that all cliques in Noetherian rings are countable. We first discuss the latter fact, beginning with a technical lemma.

Lemma 6.3. *Let R be a Noetherian ring, let $Q \in \operatorname{Spec} R$, and let $Y \subseteq \operatorname{Spec} R$. Suppose that for every $T \in \operatorname{Spec} R$ with $T \subset Q$, there is a neighborhood $\mathcal{U}_T$ of T such that there does not exist a $P \in \mathcal{U}_T \cap Y$ with $Q/T \rightsquigarrow P/T$. Then $\{ P \in Y \mid Q \rightsquigarrow P \}$ is finite.*

Proof. Suppose the lemma does not hold. By Noetherian induction we may suppose the lemma holds in any proper factor ring of R. By passing to an infinite subset of Y we may suppose that each $P \in Y$ is linked to Q via some ideal $A(P)$. Since such a link remains in any factor ring of R by an ideal contained in $A(P)$, the inductive hypothesis implies that any infinite subset of $\{ A(P) \mid P \in Y \}$ has zero intersection.

Now let T be an ideal of R which is maximal with respect to the property that it is an intersection of infinitely many elements of Y. It is not hard to see that such a T must be prime. Again, by passing to an infinite subset of Y, we may assume $T = \cap Y$. Note that $QT \subseteq \bigcap\{ QP \mid P \in Y \} \subseteq \bigcap\{ A(P) \mid P \in Y \} = 0$. Furthermore, $Q \cap T$ embeds in the direct product of the modules $Q \cap P/A(P)$, and hence is torsionfree as a left R/Q-module.

We will now show $Q \cap T = 0$. Suppose this is not true and let B be a nonzero ideal contained in $Q \cap T$ with the property that the left uniform dimension of B is as small as possible among such ideals. Then the torsionfreeness of $Q \cap T$ implies $\text{l-ann}(B) = Q$, but the uniform dimension condition implies $\text{l-ann}(B') \supset Q$ for any proper bimodule factor B' of B. For any $A(P)$, we have $B/(B \cap A(P)) \cong (B + A(P))/A(P)$, and the latter bimodule embeds in $Q \cap P/A(P)$. Thus either $B \subseteq A(P)$ or $B \cap A(P) = 0$. Further, if $B \cap A(P) = 0$, then B embeds in $Q \cap P/A(P)$ and hence $\text{r-ann}(B) = P$. Thus for all but at most one $P \in Y$, we have $B \subseteq A(P)$. The remarks in the first paragraph now imply $B = 0$. This contradiction shows that $Q \cap T = 0$ and hence R is a semiprime Noetherian ring.

Because Q is linked to other prime ideals, we cannot have $Q = 0$. On the other hand, every neighborhood of T must contain some element of Y (since $T = \cap Y$), whence the hypothesis of the lemma implies that $T \neq 0$. Thus Q and T are the distinct minimal prime ideals of R. Since 0 is a classically localizable semiprime ideal, the set $\{Q, T\}$ is link-closed, contradicting the infinitude of Y. (Alternatively: every nonminimal element of Y contains a regular element but Q doesn't, contradicting Lemma 2.4.) ∎

Corollary 6.4. *If R is a Noetherian AR-separated ring, then there are only finitely many prime ideals of R linked to any given prime ideal of R.*

Proof. Combine the lemma with the remarks before Proposition 3.2. ∎

We are now ready to prove two facts about cliques, and we then give an example of Stafford showing that the next proposition fails if we do not assume X satisfies the generic regularity condition.

Proposition 6.5. *Let R be a Noetherian ring, let X be a subset of* $\operatorname{Spec} R$ *satisfying the generic regularity condition, and let $Q \in \operatorname{Spec} R$. Then there are only finitely many prime ideals $P \in X$ with either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$.*

Proof. By symmetry, it is enough to show $\{ P \in X \mid Q \rightsquigarrow P \}$ is finite. To begin, suppose $P \in \operatorname{Spec} R$ and $Q \rightsquigarrow P$ and $Q \not\subseteq P$. Then there is an exact sequence $0 \to Q \cap P/QP \to Q/QP \to Q/Q \cap P \to 0$ of finitely generated right R/P-modules. Thus

$$\hat{\rho}(Q, P) = \hat{\rho}(Q/QP, P) = \hat{\rho}(Q \cap P/QP, P) + \hat{\rho}(Q/Q \cap P, P) = \\ = \hat{\rho}(Q \cap P/QP, P) + \hat{\rho}(Q + P/P, P) = \hat{\rho}(Q \cap P/QP, P) + 1 > 1.$$

Suppose R is prime and $Q \neq 0$. Then it is clear that $\hat{\rho}(Q, 0) = 1$ and so the Continuity Theorem implies there is a neighborhood $\mathcal{U}$ of 0 such that for any $P \in \mathcal{U} \cap X$, we have $\hat{\rho}(Q, P) = 1$. Thus if $Y_1 = \{ P \in X \mid Q \not\subseteq P \}$, no element of $\mathcal{U} \cap Y_1$ is linked to Q.

Now suppose $T \in \operatorname{Spec} R$ and $T \subset Q$. Then by passing to R/T and applying the results of the last paragraph, we get a neighborhood $\mathcal{U}_T$ of T which satisfies the hypotheses of the lemma (with Y_1 in place of Y). Thus $\{ P \in Y_1 \mid Q \rightsquigarrow P \}$ is finite by Lemma 6.3.

Set $Y_2 = \{ P \in X \mid Q \rightsquigarrow P \text{ and } Q \subseteq P \}$. We may apply Lemma 6.3 by using the neighborhood $\mathcal{U}_T = \{ P \in \operatorname{Spec} R \mid T \subseteq P \text{ and } Q \not\subseteq P \}$ and conclude that Y_2 is finite. Combining these results, we see that $\{ P \in X \mid Q \rightsquigarrow P \}$ is finite. ∎

Proposition 6.6. *If R is a Noetherian ring, every clique in R is countable.*

Proof. First we show every prime ideal in R is linked to only countably many prime ideals in R. To see this, let Q be a prime ideal in R and let X_n be the set of prime ideals P in R for which the uniform dimension of R/P is at most n. Then the set X_n satisfies the generic regularity condition by Proposition 6.2, so by Proposition 6.5, Q can be linked to only finitely many prime ideals in X_n. But every prime ideal in R lies in one of the X_n, so Q can be linked to only countably many prime ideals in R.

Now let $X = \operatorname{cl}(Q)$ where Q is a prime ideal of R. By definition of clique, any element of X can be joined to Q by a finite chain of linked primes. Since any prime can only be linked to countably many other primes, it follows that X must be countable. ∎

Example 6.7. Let k be the algebraic closure of a finite field and let G be the group generated by x, y, z subject to the relations $xy = yx$, $zxz^{-1} = xy$, $zyz^{-1} = x^2y$, so G is polycyclic-by-finite. Set $S = kG$, and let $R = \mathbb{I}_S((z-1)S) = k + (z-1)S$. Then R is a Noetherian domain in which $(z-1)S$ is a maximal ideal and $\{\, Q \in \operatorname{Spec} R \mid Q \rightsquigarrow (z-1)S \,\}$ is infinite. Moreover, the clique of $(z-1)S$ in R satisfies the second layer condition.

We now turn to the question of when generic regularity implies the intersection condition. It may be that it always does; whether this is so remains an open question. The best results now known state that this is true if R contains a large field in its center.

Theorem 6.8. *Suppose R is a Noetherian ring containing a field F in its center (or more generally a set F of central units such that the difference of any two distinct elements of F is a unit) and X is a subset of* $\operatorname{Spec} R$ *with $|X| < |F|$. If X satisfies the generic regularity condition, then X satisfies the intersection condition.*

Proof. If X is finite we know X satisfies the intersection condition, and so we may assume X and F are infinite. Let I be a right ideal of R such that $I \cap \mathcal{C}(P)$ is nonempty for every $P \in X$. We must show $I \cap \mathcal{C}(X)$ is nonempty. First we show that the set Y of prime ideals P of R for which $I \cap \mathcal{C}(P)$ is nonempty is a patch-closed subset of $\operatorname{Spec} R$. Let Q be a prime ideal of R which is an intersection of ideals of Y: we need to show that $I \cap \mathcal{C}(Q)$ is nonempty. Let I' be the right ideal $I + Q/Q$ of the ring R/Q and let J be the left annihilator of I' in R/Q. If $P \in Y$ and $P \supseteq Q$, then since $I \cap \mathcal{C}(P)$ is nonempty, we see that $J \subseteq P/Q$. By choice of Q, the intersection of all such P/Q in R/Q is 0, and hence $J = 0$. This implies I' is an essential right ideal of R/Q, so $I \cap \mathcal{C}(Q)$ is nonempty.

Thus we have shown that Y is a closed subset of $\operatorname{Spec} R$ and hence is compact. The generic regularity condition implies that for each $P \in Y$, there is a patch-open neighborhood $\mathcal{U}$ of P such that $I \cap \mathcal{C}(X \cap \mathcal{U})$ is nonempty. The compactness of Y now yields a finite collection $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of open sets in $\operatorname{Spec} R$ which cover Y and such that there exists $d_k \in I \cap \mathcal{C}(X \cap \mathcal{U}_k)$ for each k. Set $D = \{\, d_1, \ldots, d_n \,\}$ and note that since $X \subseteq Y$, the set $D \cap \mathcal{C}(P)$ is nonempty for each $P \in X$.

We now wish to show there exist $\alpha_1, \ldots, \alpha_n \in F$ such that $\alpha_1 d_1 + \cdots + \alpha_n d_n \in \mathcal{C}(X)$. Since F is infinite and $|F| > |X|$, it is enough by induction to show the following: whenever P is a prime ideal of R and r, s are elements of R such that one of them is in $\mathcal{C}(P)$, then $r + \alpha s \in \mathcal{C}(P)$ for all but a finite number of choices of $\alpha \in F$. (This tells us that to get elements regular modulo each $P \in X$, we can take combinations of $d_1, \ldots, d_n$ with coefficients in F, except that for each P, we must throw out finitely many possible choices of coefficients. Since $|F| > |X|$, this is O.K. .)

We now prove the required fact. First of all we may pass to R/P and so assume $P = 0$. Since F is a central subfield, multiplying by nonzero elements of F (such as α or $\alpha - 1$) does not change $\mathcal{C}(P)$, and so we may assume without loss of generality that $s \in \mathcal{C}(0)$. For each $\alpha \in F$, define $K_\alpha = \text{r-ann}(r + \alpha s)$. It is not hard to see (using regularity of s and ideas like those in the proof of the linear independence of eigenvectors corresponding to different eigenvalues) that any sum of right ideals K_α with distinct indices is a direct sum. Since R has finite uniform dimension, this means only finitely many K_α can be nonzero. For all but these values of α, then, $r + \alpha s$ is a regular element of R. ∎

Corollary 6.9. *Let R be a Noetherian ring with center Z and let X be a clique in R satisfying the second layer condition and the generic regularity condition. If either Z contains an uncountable subfield or $Z/Z \cap P$ is uncountable for $P \in X$, then X is classically localizable.*

Proof. In the former case, let F be the hypothesized subfield of Z. In the latter case, we can localize R at $Z \setminus Z \cap P$ (since $Z \setminus Z \cap P \subseteq \mathcal{C}(X)$) and so assume Z is local. If we now take F to be a set with one representative from each nonzero coset of the residue field of Z, then F is uncountable and the difference of any two distinct elements of F is a unit. Now apply Theorems 5.4, 6.6, and 6.8. ∎

I know of no example of a clique in a Noetherian ring satisfying the second layer condition which is not classically localizable, and I know of no example of a clique satisfying one of the generic regularity condition or the intersection condition which does not satisfy the other.

I would like to mention without proof one other result concerning the intersection condition. Suppose R is an algebra over a field of characteristic 0. We say a set X of completely prime ideals in R satisfies $(\#\cap)$ if for any finite subset A of R, there is a positive integer s such that for any irreducible polynomial $\sum_{i=0}^{t} \alpha_i x^i$ in $\mathbb{Z}[x]$ of degree $t > s$, any $P \in X$, and any $a, b \in A \setminus P$, we have $\sum_{i=0}^{t} \alpha_i a^i b^{t-i} \notin P$. Note than a finite union of sets satisfying $(\#\cap)$ satisfies $(\#\cap)$.

Proposition 6.10 (Goodearl)**.** *If R is a right Noetherian algebra over a field of characteristic 0 and X is a set of completely prime ideals in R satisfying $(\#\cap)$, then X satisfies the right intersection condition.*

We now summarize the known results on localizable cliques in Noetherian rings. The rings mentioned satisfy the second layer condition, and the generic regularity condition and/or the intersection condition can be verified for cliques in these rings, using for example finiteness of cliques in the nilpotent cases and many of the other rings listed, the result of Sigurdsson mentioned in Appendix A in the Lie algebra cases, and the result of Brown mentioned in Appendix B in the group ring case. For details see the references. In the case of (1) and (9), note that the "uncountable central subfield" hypothesis has been dropped. These results are due to Müller and Goodearl respectively, and require other methods.

Theorem 6.11. *Every clique in the following rings is classically localizable.*

(1) *Noetherian p.i. rings which are finitely generated algebras over their centers.*
(2) *Artinian rings.*
(3) *Principal ideal rings.*
(4) *Hereditary Noetherian Prime rings with enough invertible ideals.*
(5) *Noetherian fully bounded algebras (e.g., p.i. algebras) over an uncountable field.*
(6) *Group rings RG where G is a polycyclic-by-finite group and R is a commutative Noetherian ring such that all prime factor rings of R either have positive characteristic or are uncountable, or G is finitely generated nilpotent and R is commutative Noetherian.*

(7) *Enveloping algebras* $R \otimes_k U(\mathfrak{g})$ *where* k *is a field,* R *is a commutative Noetherian* k*-algebra,* $\mathfrak{g}$ *is a finite dimensional Lie* k*-algebra, and if* k *has characteristic zero, then* $\mathfrak{g}$ *is solvable.*
(8) *Enveloping algebras* $U(\mathfrak{g})$ *where* $\mathfrak{g}$ *is a finite-dimensional Lie superalgebra over a field* k*, and if* k *has characteristic zero, then* $\mathfrak{g}$ *is solvable and* k *is uncountable.*
(9) *Skew enveloping algebras* $R\#_\sigma U(\mathfrak{g})$ *where* k *is a field of characteristic zero,* R *is a commutative Noetherian* k*-algebra, and* $\mathfrak{g}$ *is a finite-dimensional solvable Lie* k*-algebra acting on* R *as derivations.*
(10) *Ore extensions* $R[x, x^{-1}; \phi]$, $R[x; \phi]$, $R[\theta; \delta]$ *where* R *is a commutative Noetherian algebra over an uncountable field* k, ϕ *is a* k*-linear automorphism of* R, *and* δ *is a* k*-linear derivation on* R. ∎

7. Rings with the second layer condition and finite extensions

We say a ring satisfies the *right second layer condition* if all of its prime ideals do. In this section we discuss some properties of such rings, and we also discuss some results concerning finite extensions of rings. The results in this section give a hint as to how the methods in these notes can be applied to the study of questions in Noetherian ring theory that do not involve localization.

Recall that a *Noetherian bimodule over* R is an R-R-bimodule which is Noetherian as both a right and a left module. Using the Noetherianness on the "wrong" side, it is a standard result that if B is a Noetherian bimodule, $R/\operatorname{r-ann}(B)$ embeds as a right module in a finite direct sum of copies of B and that if $\operatorname{r-ann}(B)$ is prime, B is torsionfree as a right $R/\operatorname{r-ann}(B)$ module. The corresponding results on the left are true as well.

Lemma 7.1. *Suppose* R *is a Noetherian ring satisfying the right second layer condition and* B *is a Noetherian bimodule over* R. *If* P *is a prime ideal in* R *such that* $P \supseteq \operatorname{r-ann}(B)$, *then there is a bimodule subfactor* C *of* B *and a prime ideal* Q *in* R *such that* $\operatorname{l-ann}(C) = Q$, $\operatorname{r-ann}(C) = P$, *and* C *is torsionfree as both a left* R/Q*- and right* R/P*-module.*

Proof. First we make a reduction. Let B_1' be a nonzero subbimodule of B with $P_1 = \operatorname{r-ann}(B_1')$ as big as possible, and then let B_1 be a nonzero subbimodule of B_1' with $Q_1 = \operatorname{l-ann}(B_1)$ as big as possible. Then standard arguments show Q_1, P_1 are prime ideals and B_1 is torsionfree as both a left R/Q_1- and right R/P_1-module. Repeating this argument we see there is a chain $0 = B_0 < B_1 < \cdots < B_n = B$ of subbimodules such that $Q_i = \operatorname{l-ann}(B_i/B_{i-1})$ and $P_i = \operatorname{r-ann}(B_i/B_{i-1})$ are prime and B_i/B_{i-1} is torsionfree as both a left R/Q_i- and right R/P_i-module. Clearly $P \supseteq P_n \cdots P_1$, and so P contains some P_i. Thus by passing to a subfactor of B, we may assume $\operatorname{r-ann}(B) = P'$ and $\operatorname{l-ann}(B) = Q'$ are prime and B is torsionfree as both a left R/Q'- and right R/P'-module. If $P = P'$ we're done.

Suppose $P \supset P'$ and let E be the injective hull of R/P as a right R/P'-module. Since R/P' embeds in a direct sum of copies of B as a right module, the natural map from R/P' to R/P to E gives rise to a nonzero map $f : B \to E$. Clearly $\operatorname{ass} f(B) = P$ and so by Lemma 4.3 we have $f(B)P_n \cdots P_1 = 0$ where the primes P_i contain P' and each P_i/P' is in the right clique of P/P' in R/P', and in addition $P_1 = P$. Since no prime can be linked to 0, each $P_i \supset P'$ and so if $I = \operatorname{ann} f(B)$, we have $P \supseteq I \supset P'$.

If $J = \text{r-ann}(B/BI)$, then $f(B)J = f(BJ) \subseteq f(BI) = 0$, so $\text{r-ann}(B/BI) = I$. It follows by Noetherian induction that the bimodule B/BI has a subfactor of the required type. ∎

We define the *classical Krull dimension* of a ring R as follows. We let $\text{Spec}^0(R)$ be the set of maximal ideals. For an arbitrary ordinal α, we define $\text{Spec}^\alpha(R)$ to be

$$\{\, P \in \text{Spec}\, R \mid \text{ for all } Q \in \text{Spec}\, R \text{ with } Q \supset P,\, Q \in \text{Spec}^\beta(R) \text{ for some } \beta < \alpha \,\}.$$

It is clear that $\text{Spec}\, R = \text{Spec}^\alpha(R)$ for some ordinal α. We define $\text{cl Krull dim}\, R$ to be the smallest such α. If R is right Noetherian, for every ordinal $\beta \leq \text{cl Krull dim}\, R$, there is a prime ideal $P \in \text{Spec}^\beta(R)$ with $\text{cl Krull dim}\, R/P = \beta$. If Q, P are prime ideals of R with $P \supset Q$, then $\text{cl Krull dim}\, R/P < \text{cl Krull dim}\, R/Q$.

Theorem 7.2 (Jategaonkar)**.** *If R is a Noetherian ring satisfying the second layer condition and B is a Noetherian bimodule over R, then*

$$\text{cl Krull dim}\, R/\,\text{l-ann}(B) = \text{cl Krull dim}\, R/\,\text{r-ann}(B).$$

Proof. We show $\text{cl Krull dim}\, R/\,\text{l-ann}(B) \geq \text{cl Krull dim}\, R/\,\text{r-ann}(B)$. We proceed by induction on $\alpha = \text{cl Krull dim}\, R/\,\text{r-ann}(B)$. If $\alpha = 0$, the inequality is clear, so suppose $\alpha > 0$. There is a prime ideal $P \supseteq \text{r-ann}(B)$ with $\text{cl Krull dim}\, R/P = \alpha$. Choose a bimodule subfactor C of B and a prime ideal Q of R as in Lemma 7.1. Then $Q \supseteq \text{l-ann}(B)$ and so $\text{cl Krull dim}\, R/Q \leq \text{cl Krull dim}\, R/\,\text{l-ann}(B)$. Thus we may replace B by C and so assume $\text{l-ann}(B) = Q$, $\text{r-ann}(B) = P$, and B is torsionfree as both a left R/Q- and right R/P-module.

Suppose $\beta < \alpha$ and let $P' \supset P$ be a prime ideal of R such that $\text{cl Krull dim}\, R/P' = \beta$. Let B' be a nonzero subbimodule of B whose left uniform dimension is as small as possible. Apply Lemma 7.1 again to find a bimodule subfactor C' of B' and a prime ideal Q' of R such that $\text{l-ann}(C') = Q'$, $\text{r-ann}(C') = P'$, and C' is torsionfree as both a left R/Q'- and right R/P'-module. This C' cannot be a subbimodule of B' since $P' \supset P$. Thus it is a factor of a subbimodule by another subbimodule of the same left uniform dimension. It follows that C' is torsion as a left R/Q-module, and so by standard bimodule arguments, $Q' = \text{l-ann}(C') \supset Q$. We thus have by induction on α that $\text{cl Krull dim}\, R/Q > \text{cl Krull dim}\, R/Q' \geq \text{cl Krull dim}\, R/P' = \beta$. Since this is true for any $\beta < \alpha$, we conclude that $\text{cl Krull dim}\, R/Q \geq \alpha$, proving the theorem. ∎

Corollary 7.3. *If R is a Noetherian ring satisfying the second layer condition and Q, P are prime ideals of R in the same clique, then* $\text{cl Krull dim}\, R/Q = \text{cl Krull dim}\, R/P$*, and so Q and P are either incomparable or equal.* ∎

Corollary 7.4. *If R is a Noetherian ring satisfying the second layer condition and Q, P are prime ideals of R with* $\text{l-ann}(Q \cap P/QP) = Q$ *and* $\text{r-ann}(Q \cap P/QP) = P$*, then $Q \rightsquigarrow P$.*

Proof. Let T be the right R/P-torsion submodule of $Q \cap P/QP$, and note T is a subbimodule. By standard bimodule arguments, $\text{r-ann}(T/QP) \supset P$, and so $Q \cap P/T \neq 0$. As $Q \cap P/T$ is torsionfree over R/P, we have $\text{r-ann}(B') = P$ for any nonzero subbimodule B' of $Q \cap P/T$. If $J = \text{l-ann}(B')$, then $J \supseteq Q$ and $\text{cl Krull dim}\, R/J = \text{cl Krull dim}\, R/P = \text{cl Krull dim}\, R/Q$, so $J = Q$. It follows that $Q \rightsquigarrow P$. ∎

The next result shows that in a Noetherian ring satisfying the second layer condition, cliques defined by either second layer links or ideal links are the same thing. Borho has shown this is also true in the enveloping algebra of a complex semisimple Lie algebra, but the example after the proposition shows that it is not always true.

Proposition 7.5 (Jategaonkar)**.** *If R is a Noetherian ring satisfying the second layer condition and Q, P are prime ideals of R such that there is an ideal link from Q to P, then $Q \in \operatorname{rt}\operatorname{cl}(P)$.* ∎

Example 7.6 (Stafford)**.** Let $\Omega = h^2 - 2h + 4ef$ be the Casimir element in $U = U(sl_2(\mathbb{C}))$ (Ω is central) and let $M = eU + fU + hU$ be the augmentation ideal. Set $S = eM + \mathbb{C}[h] \subset eU + \mathbb{C}[h] = \mathbb{I}_U(eU)$, and note $\Omega \in S$. Then $R = S/\Omega S$ is a Noetherian ring with exactly two maximal ideals P_1, P_2 (the images of $eM + (h-2)\mathbb{C}[h]$ and $eM + h\mathbb{C}[h]$), one nilpotent prime ideal N (the image of $\Omega eU + \Omega S$), and one other nontrivial ideal ($P_1 \cap P_2 = P_1P_2 = P_2P_1$). The cliques in R are all singletons, but P_1 and P_2 are ideal-linked via N.

The next two results and the example afterward show that one can frequently but not always move the second layer condition up or down a finite ring extension. An interesting related question is still open: if R is a Noetherian ring satisfying the second layer condition, does the polynomial ring $R[x]$ satisfy it?

In the next two results we use the reduced rank, which is related to the normalized rank introduced before (and which is used in the standard proof of Small's Theorem). We will not give its definition and will freely use its basic properties; we note here two of those properties. Reduced rank is additive on short exact sequences, and if $R\mathcal{C}_R(0)^{-1}$ exists and is Artinian, then a finitely generated R-module has reduced rank 0 if and only if it is $\mathcal{C}_R(0)$-torsion.

Lemma 7.7. *Suppose R and S are Noetherian rings and R is a subring of S. Suppose in addition that every element of S is contained in a subring $S' \supseteq R$ such that S' is finitely generated as both a right and left R-module.*

(a) If $\mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$ and $R\mathcal{C}_R(0)^{-1}$ exists and is Artinian, then $S\mathcal{C}_S(0)^{-1}$ exists and is Artinian.

(b) If S is prime and R satisfies the second layer condition, then $\mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$ and $R\mathcal{C}_R(0)^{-1}$ exists and is Artinian.

Proof. (a) Since $\mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$, if $c \in \mathcal{C}_R(0)$, we have that $cS \cong S$ as right R-modules, and so S/cS has reduced rank 0. Thus for every $s \in S$, there is a $d \in \mathcal{C}_R(0)$ with $(s + cS)d = 0$. It follows that $\mathcal{C}_R(0)$ is a right Ore set in S. Thus we may localize at $\mathcal{C}_R(0)$, and so assume R is Artinian. If N is the prime radical of S and $s + N$ is regular in S/N, then let S' be a subring of S containing s such that S' is finitely generated as an R-module. Note that S' is Artinian. Now $S' + N$ is a subring of S, and we see that $(S' + N)/N$ is Artinian. Clearly $s + N$ is regular in $(S' + N)/N$, and so it is a unit in $(S' + N)/N$, which of course implies it is a unit in S/N. Since S/N is a semiprime Noetherian ring, it follows that S/N is Artinian. From this we conclude that S is Artinian.

(b) Let Q be the Goldie quotient ring of S, let $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_k = Q$ be a composition series for Q as a Q-R-bimodule, and let $P_i = \text{r-ann}(Q_i/Q_{i-1})$. It follows from work of Warfield on the additivity principle that the P_i's are precisely the

minimal prime ideals of R. Let $N = \cap P_i$ be the prime radical of R and suppose $c \in \mathcal{C}_R(N) = \bigcap \mathcal{C}_R(P_i)$. Since each Q_i/Q_{i-1} is torsionfree over R/P_i, working down the composition series shows that if $qc = 0$ for $q \in Q$, then $q = 0$. Thus c is left regular in the Artinian ring Q, and so c is a unit in Q, and hence is regular in S. It follows that $\mathcal{C}_R(N) \subseteq \mathcal{C}_S(0) \cap R \subseteq \mathcal{C}_R(0)$. By Small's Theorem, equality holds throughout and $R\mathcal{C}_R(0)^{-1}$ exists and is Artinian. ∎

Theorem 7.8. *Suppose R and S are Noetherian rings and R is a subring of S.*

(a) [Letzter] *If S is finitely generated as a right and left R-module and R satisfies the [strong] second layer condition, then S satisfies the [strong] second layer condition.*

(b) Suppose that every element of S is contained in a subring $S' \supseteq R$ such that S' is a finitely generated free R-module with a basis which centralizes R. If S satisfies the [strong] second layer condition, then R satisfies the [strong] second layer condition.

Proof. We proof these results for the "plain" second layer condition.

(a) Suppose S does not satisfy the right second layer condition, and let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated uniform right S-modules as in the Main Lemma with $L \subseteq S/P$, $\operatorname{ann}(M) = \operatorname{ann}(N) = Q \subset P$ for prime ideals Q, P of S. By passing to $R/Q \cap R \subseteq S/Q$, we may assume $Q = 0$. The last lemma implies that S/P, and hence L, is torsionfree as a right $R/P \cap R$-module (i.e., with respect to $\mathcal{C}_R(P \cap R)$) and (by Small's Theorem) that a prime of R is minimal over $P \cap R$ if and only if it does not contain an element of $\mathcal{C}_R(P \cap R)$. Suppose W is an R-submodule of L and $\operatorname{ann}(W) = P'$ for a prime ideal P' of R containing $P \cap R$. A standard argument using torsionfreeness shows that P' is minimal over $P \cap R$ and W is torsionfree as an R/P'-module.

Choose an R-submodule W of M maximal with respect to $L \cap W = 0$. Then L is an essential R-submodule of M/W, and the last paragraph along with Lemma 4.3 implies that there are primes $Q_1, \ldots, Q_n$ in R which are connected by links to the minimal primes of $P \cap R$ and such that $(M/W)Q_n \cdots Q_1 = 0$. Since S is prime, P contains a regular element c of S, so P contains the isomorphic copy cS of S. It follows that the reduced rank of S/P as an R-module is 0, and so P contains a regular element of R. Thus by Lemma 2.4, every prime ideal Q_i contains a regular element of R, and so $J = \operatorname{ann}_R(M/W)$ contains a regular element. Hence S/SJ is torsion as a left S-module, which implies $I = \text{l-ann}_S(S/SJ)$ is a nonzero ideal of S, since S/SJ is finitely generated as a right R-module. Thus $MI = MIS \subseteq MSJ = MJ \subseteq W$. But then MI is an S-submodule of M with $MI \cap L = 0$, so $MI = 0$. This contradicts the fact that $\operatorname{ann}(M) = 0$.

(b) Suppose R does not satisfy the right second layer condition, and let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated uniform right R-modules as in the Main Lemma with $L \subseteq R/P$, $\operatorname{ann}(M) = \operatorname{ann}(N) = Q \subset P$ for prime ideals Q, P of R. By passing to $R/Q \subseteq S/QS$, we may assume $Q = 0$. (It is clear that $QS \cap R = Q$ and that the hypotheses are still valid.) The local freeness of the extension implies that $\mathcal{C}_R(0) \subseteq \mathcal{C}_S(0)$, and so Lemma 7.7(a) applies. It also implies $0 \to L \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$ is exact and that $L \otimes_R S$ is essential in $M \otimes_R S$. The local freeness with a centralizing basis implies $\operatorname{ann}_S(L \otimes_R S) = PS$ and $\operatorname{ann}_S(M \otimes_R S) = 0$. Now $L \otimes_R S$ embeds in $(R/P) \otimes_R S \cong S/PS$. Since PS is an ideal of S and S is Noetherian, a standard argument shows that if W is an S-submodule of S/PS with

$Q' = \operatorname{ann}_S(W)$ prime, then W is torsionfree as an S/Q'-module. It follows that the same is true of any S-submodule of $M \otimes_R S$. Furthermore, $Q' \cap R \supseteq P$ and so Q' contains a regular element of R, and hence of S.

Applying Lemma 4.3 to $M \otimes_R S$, we see that $(M \otimes_R S)Q_n \cdots Q_1 = 0$ for some prime ideals Q_i in S, each of which is in the clique of some such Q'. By Lemma 2.4, we see each Q_i contains a regular element of S, and so $\operatorname{ann}(M \otimes_R S)$ contains a regular element, contrary to the above. ∎

Remark . Suppose R is a Noetherian k-algebra and E is a field extension of k. Then Theorem 7.8 tells us that if E/k is finite and R satisfies the second layer condition, then $R \otimes_k E$ satisfies the second layer condition, while if E/k is algebraic and $R \otimes_k E$ is Noetherian and satisfies the second layer condition, then R satisfies the second layer condition.

Example 7.9 (Hodges-Osterburg)**.** Let k be a field of characteristic 2 and let λ be an element of k which is not a root of unity. Let S be the k-algebra which is generated by $x^{\pm 1}, y^{\pm 1}$ subject to the relation $xy = \lambda yx$ and let σ be the k-linear automorphism of S defined by $\sigma(x) = x^{-1}, \sigma(y) = y^{-1}$, so σ has order two. Set $R = S^\sigma = \{ s \in S \mid \sigma(s) = s \}$. Then R and S are both Noetherian domains of Krull dimension 1, and S is a finitely generated projective R-module on both the right and the left. Moreover, S is simple and so satisfies the second layer condition, while R has exactly two proper ideals, 0 and a maximal ideal P with $P^2 = P$. Such a P cannot satisfy the second layer condition.

To verify this last claim, note that R_R embeds in P_R, so if $E = E(R/P)$, there is a nonzero map $f : P \to E$. Since $0 \neq f(P) = f(PP) = f(P)P$, we have $\operatorname{ann} f(P) = 0$. Shrinking $f(P)$ if necessary to get M, we can obtain a short exact sequence $0 \to L \to M \to N \to 0$ as in the Main Lemma with $\operatorname{ann} M = \operatorname{ann} N = 0$ and $L \subseteq R/P$. Thus P does not satisfy the second layer condition.

Letzter and others have used the techniques described here to obtain many interesting results about finite extension rings, including results on lying over which are sensitive to links and to primitivity. We mention just two results. One is a result of Goodearl which states that if R is a Noetherian ring satisfying the second layer condition and S is a finite centralizing extension of R which is flat on both sides, then for any clique X in R, there is a clique Y in S with $X = \{ P \cap R \mid P \in Y \}$. The other is the following.

Theorem 7.10 (Letzter)**.** *Suppose R is a Noetherian ring satisfying the right second layer condition and S is an overring of R which is finitely generated as both a right and left R-module. If there is a finite bound on the uniform dimensions of prime factor rings of R, then there is a finite bound on the uniform dimensions of prime factor rings of S.* ∎

I would like to mention a result of Warfield. He shows that in a Noetherian ring satisfying the second layer condition, if the intersection of the prime ideals in a classically localizable clique is a prime ideal, then that prime ideal is classically localizable. More generally, if the intersection is not prime, the set of primes of maximal classical Krull codimension among the primes minimal over the intersection forms a classically localizable set.

8. Notes on references

The main reference for the material in these notes is A. V. Jategaonkar's book *Localization in Noetherian rings* ([55]), which contains proofs of most of the results stated here along with a great deal more. A nice brief introduction to this material is K. A. Brown's *Ore sets in Noetherian rings* ([16]). Most of these ideas were developed by Jategaonkar, including the second layer condition and the intersection condition. Brown also proved many of these results, independently of Jategaonkar. For a short historical discussion, see R. B. Warfield, Jr.'s review of Jategaonkar's book ([102]). Some of the technical results related to Goldie's Theorem and localization, as well as general facts about Noetherian rings, appear in A. W. Chatters' and C. R. Hajarnavis' book [31], J. C. McConnell's and J. C. Robson's book [72], and K. R. Goodearl and Warfield's book [47]. These books also contain proofs of Small's Theorem. The proof of Lemma 1.11 can be found in the papers [93] and [97] by L. W. Small and J. T. Stafford.

The second layer condition (originally called "condition $(*)$") was introduced – and many properties of rings in which all prime ideals satisfy it were determined – by Jategaonkar in [53] and [54]. In the latter paper he showed that all prime ideals in group rings of polycyclic-by-finite groups over commutative Noetherian rings, universal enveloping algebras of finite dimensional solvable Lie algebras, HNP rings with enough invertible ideals, and FBN rings satisfy the second layer condition. Brown proved similar results in the group ring and enveloping algebra cases, and he also studied the structure of links and cliques in these rings in several papers, including [12], [13], [14] (contains the proof of Theorem A.2), [15], [19], and [21] (with F. Du Cloux). Brown proved Theorem 6.11(f) in [17]. See also Goodearl's discussion of cliques in enveloping algebras and in (generalizations of) smash products in [38], where Theorem 6.11(i) is proved. Gunnar Sigurdsson has also studied cliques in enveloping algebras in [92]. W. Borho discusses links in enveloping algebras of complex semisimple Lie algebras in [4]. For a proof that all prime ideals in some more general types of rings satisfy the second layer condition, including primes in certain smash products, skew enveloping algebras, and group-graded rings, see A. D. Bell's papers [2] and [1].

E. S. Letzter proves Theorems 7.8(a) and 7.10 in [64], and he also studies the relationship between links in R and S, where S is finitely generated as an R-module on both sides. He shows, for example, that if all primes in R and S satisfy the second layer condition, then if two primes are linked in R, there are primes lying over them which are equal or are connected by a sequence of links. This does not in general imply that cliques of R can be lifted to cliques of S; Goodearl's result that this can be done when S is also a centralizing extension of R which is flat on both sides appears in [38]. Letzter uses his results to get information on links and localizability of cliques in enveloping algebras of Lie superalgebras, showing that the results parallel those for ordinary Lie algebras. (Letzter also shows that a finite extension of an FBN ring is FBN.) Example 7.9 of T. J. Hodges and J. Osterburg appears in [48]. The work of Warfield on the additivity principle cited in the proof of Lemma 7.7 can be found in [100] or in [72].

Another key investigator in this subject has been B. J. Müller, who did early work on localizing at semiprime ideals. For example, in [74] he showed that the cliques in a Noetherian ring which is a finitely generated module over its center are the sets of primes lying over single prime ideals in the center, that all cliques are finite, and that the localization at a clique can be achieved by inverting only central elements. His paper

[79] shows that all cliques in affine Noetherian p.i. rings are classically localizable. This is one of the few results for infinite cliques which drops the "uncountable central subfield" hypothesis. Theorem 5.8 of A. Braun and Warfield for p.i. rings is found in [11]. (In [8] Braun has studied a situation where the localization theory can be extended to affine p.i. rings without the Noetherian hypothesis.) In [61], T. H. Lenagan and Letzter prove some facts about primes occurring in the "layers" of injective hulls of modules over Noetherian p.i. rings, as well as proving results on lying over that are sensitive to ideal links.

Some of the key ideas in Section 6 are due to Warfield in [101] and Stafford in [99]. Warfield also studies the localized ring R_X, proving for example that under the hypotheses of Theorem 6.8, the stable rank of R_X is 1, finitely generated projective modules are determined up to isomorphism by their normalized rank at primitive ideals, and finitely generated projectives can be cancelled. Stafford provides a proof of Propositions 6.5 and 6.6. He gives Examples 6.7 and 7.6 in [98]. Proposition 6.10 was proved by Goodearl in [38]. The result of Warfield mentioned at the end of Section 7 appears in [103].

The generic regularity condition was introduced by Stafford and Warfield. For a discussion of it, as well as a proof of the continuity theorem of Stafford stated in Section 6 and a discussion of some of its consequences (including Proposition 6.2), see Goodearl's paper [36] (see also [55] or [72]). Goodearl has initiated an investigation of localization in the absence of the second layer condition in [37] by concentrating on injective modules. Stafford, with help from Goodearl, has proven in unpublished work that if X is a clique in a Noetherian p.i. ring R, then $\{ P[x] \mid P \in X \}$ is a localizable clique in the polynomial ring $R[x]$. The possibility of generalizing of this result adds interest to the open question of whether the second layer condition always goes up to polynomial rings – in fact Stafford has shown $\{ P[x] \mid P \in X \}$ is localizable for a general Noetherian ring R if we know $\{ P[x] \mid P \in X \}$ satisfies the second layer condition and the generic regularity condition, eliminating all uncountability hypotheses.

Some information on the Artin-Rees property is contained in the books [31] and [72], including proofs of P. F. Smith's result that $1-P$ is a right Ore set if P is an ideal with the right AR property and a version of the following result of Smith which is sometimes useful: if R is a right Noetherian ring, I an ideal of R with the right AR property, and C a multiplicatively closed set in R such that $C+I \subseteq C$ and such that the image of C is a right Ore set in R/I^n for all positive integers n, then C is a right Ore set in R. The AR property is also used by McConnell in [70] and is discussed by J. E. Roseblade in [88] and Smith in [96]. McConnell proves a result on the intersection of the powers of an ideal which overlaps with Proposition 3.7 in [70], as does Smith in [94]. For the example of a classically localizable prime ideal without the AR property cited in Section 3, see Smith's paper [95].

Links and cliques in differential operator rings $R[\theta;\delta]$, where R is a commutative Noetherian ring containing $\mathbb{Q}$ and δ is a derivation on R, have been studied by G. Sigurdsson in [91]. He is able to show directly that $\mathcal{C}(X)$ is an Ore set for cliques X in such rings and he shows that cliques consist of either a single prime ideal or a countably infinite set of prime ideals. This paper also contains an example of a clique which is strictly bigger than the union of the right and left cliques generated by one of its prime ideals. Links and cliques in skew polynomial and skew Laurent rings $R[x;\phi]$ and $R[x,x^{-1};\phi]$, where R is a commutative Noetherian ring and ϕ is an automorphism of R, have been studied by D. G. Poole in [87]. (In this case cliques may be countably

infinite or have any finite cardinality.) The result of Sigurdsson on uniform dimension noted after Proposition 6.2 is contained in [90].

A number of other papers related to localization and the second layer condition are listed in the bibliography. A number of papers have provided applications both of the localization theory itself and of the ideas associated with it. The connection between module extensions and ideal structure has proven to be particularly interesting. A few of the papers of interest not mentioned above include Brown and Warfield's *Krull and global dimensions of fully bounded Noetherian rings* ([26]), Goodearl and Small's *Krull versus global dimension in Noetherian p.i. rings* ([45]), Brown and Warfield's *The influence of ideal structure on representation theory* ([27]), Brown and T. Levasseur's *Cohomology of bimodules over enveloping algebras* ([24]), Brown's *Fully bounded Noetherian rings of finite injective dimension* ([18]), Letzter's *Primitive ideals in finite extensions of Noetherian rings* ([63]), Musson's *Injective modules, localization and completion in group algebras* ([81]), Brown's *The representation theory of Noetherian rings* ([20]), and Lenagan's and Warfield's *Affiliated series and extensions of modules* ([62]). Jategaonkar's and Letzter's results show links preserve subdirect irreducibility, semiprimitivity, and right or left primitivity of R/P.

For more information on p.i. rings, see [72]. For more information on Lie algebras and enveloping algebras, see [5], [49], and [50]. For more information on the PBW extensions discussed in Appendix A, see [38], where Theorem A.1 is proven. For more information on group rings and on the group-graded rings discussed in Appendix B, see [85] and [86].

Appendix A. Enveloping algebras and PBW extensions

In this appendix we summarize many of the known results on localization in enveloping algebras and in some generalizations thereof. Throughout this appendix, $\mathfrak{g}$ is a finite-dimensional Lie algebra (usually solvable) over a field k with enveloping algebra $U = U(\mathfrak{g})$. The key to studying enveloping algebras is the Poincaré-Birkhoff-Witt Theorem, so one generalization of enveloping algebras would be extensions S of a ring R such that S is a free R-module with the ordered monomials (those of the form $x_1^{i(1)} \cdots x_n^{i(n)}$ where $i(1), \ldots, i(n)$ are nonnegative integers) in some $x_1, \ldots, x_n \in S$ as a basis and where we also require that each $[x_i, x_j] = x_i x_j - x_j x_i$ lie in $Rx_1 + \cdots + Rx_n + R$ and that each $[x_i, r]$ lie in R. We call such a ring S a *PBW extension* of R: note that S is Noetherian if R is. This includes a smash product $R\#_\sigma U(\mathfrak{g})$ where $\mathfrak{g}$ acts on R as derivations and σ is a Lie cocycle.

If k has positive characteristic, $U(\mathfrak{g})$ is a finitely generated module over its center, and so all cliques in $U(\mathfrak{g})$ are classically localizable by Theorem 6.11(a).

If k is algebraically closed and has characteristic 0 and $\mathfrak{g}$ is solvable, then Lie's Theorem implies there is a chain $0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of ideals of $\mathfrak{g}$ with $\dim \mathfrak{g}_i = i$. Thus if $y \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$, there is a $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ (we think of such functionals as elements of $\mathfrak{g}^*$ which vanish on $[\mathfrak{g}, \mathfrak{g}]$) such that $[x + \mathfrak{g}_{i-1}, y + \mathfrak{g}_{i-1}] = \lambda(x)y + \mathfrak{g}_{i-1}$ for all $x \in \mathfrak{g}$. The eigenvalues λ that occur are independent of the chain of ideals chosen, and are called the *Jordan-Hölder values* of $\mathfrak{g}$. Of course $\mathfrak{g}$ is nilpotent if and only if each $\lambda = 0$. We will let $\Gamma(\mathfrak{g})$ denote the additive subgroup of $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ generated by the Jordan-Hölder values of $\mathfrak{g}$. One can show by induction that if A, B are ideals of U with $A \subset B$, then there is an element $b \in B \setminus A$ and a $\delta \in \Gamma(\mathfrak{g})$ such that $[x + A, b + A] = \delta(x)b + A$ for all $x \in \mathfrak{g}$. It follows that $b + A$ is a normal element

of B/A, and so it generates an ideal of U/A with the AR property. Proposition 4.6 now implies that every prime ideal in U satisfies the strong second layer condition. (The existence of this normal element can also be deduced by applying Lie's Theorem to the locally finite $\mathfrak{g}$-module B/A.) For an arbitrary k of characteristic 0, let $\bar{k}$ denote the algebraic closure of k. We have just seen that $U(\mathfrak{g} \otimes_k \bar{k}) = U(\mathfrak{g}) \otimes_k \bar{k}$ satisfies the strong second layer condition, and so by the "strong" version of Theorem 7.8(b), $U(\mathfrak{g})$ satisfies the strong second layer condition.

One can in fact show that any "solvable" (we leave it to the reader to formulate the appropriate definition) PBW extension of a commutative Noetherian algebra over a field of characteristic 0 satisfies the second layer condition, using techniques like those in Appendix B.

A theorem of Dixmier, greatly extended by Sigurdsson, shows that if $\mathfrak{g}$ is solvable and k has characteristic 0, then every prime ideal in $U(\mathfrak{g})$ is completely prime. In fact Sigurdsson's result shows that all prime ideals in "solvable" PBW extensions of commutative Noetherian algebras over fields of characteristic 0 are completely prime, and so the corresponding factor rings have uniform dimension 1. Goodearl shows that any clique in such a ring satisfies $(\#\cap)$, and so using Proposition 6.10, he is able to prove the following result.

Theorem A.1. *If R is a factor ring of a "solvable" PBW extension of a commutative Noetherian algebra over a field of characteristic 0, then all cliques in R are classically localizable.* ∎

Let us now give Brown's description of links in $U = U(\mathfrak{g})$. We assume that k has characteristic 0 and $\mathfrak{g}$ is solvable, and that Lie's Theorem applies to $\mathfrak{g}$ (e.g., k is algebraically closed). Given $\lambda \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$, we can define a *winding automorphism* τ_λ of U by $\tau_\lambda(x) = x + \lambda(x)$ for $x \in \mathfrak{g}$. Suppose Q is ideal-linked to P via B/A. Let $b + A$ be the normal element of B/A defined above. Since B/A is torsionfree on each side, we get $Q = \text{l-ann}(b + A)$, $P = \text{r-ann}(b + A)$. Since

$$x(b + A) = [x, b] + bx + A = \delta(x)b + bx + A = (b + A)\tau_\delta(x) \text{ for all } x \in \mathfrak{g},$$

we have $x(b + A) = (b + A)\tau_\delta(x)$ for all $x \in U$. Thus $P = \tau_\delta(Q)$ for this $\delta \in \Gamma(\mathfrak{g})$. A more precise description of links is given by the following result. We remark that the notation in the following theorem (e.g., $\mathfrak{h}$) depends on certain choices.

Theorem A.2 (Brown)**.** *Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra with basis $x_1, \ldots, x_n$ such that $\mathfrak{g}_i = \mathbb{C}x_1 + \cdots + \mathbb{C}x_i$ is an ideal of $\mathfrak{g}$ for all i and such that some $\mathfrak{g}_i$ is the largest nilpotent ideal of $\mathfrak{g}$. Let $\mathcal{S} = \{\, i \mid P \cap U(\mathfrak{g}_i) \neq (P \cap U(\mathfrak{g}_{i-1}))U(\mathfrak{g}_i) \,\}$, let λ_i be the eigenvalue of $\operatorname{ad}\mathfrak{g}$ on $\mathfrak{g}_i/\mathfrak{g}_{i-1}$, and let $\mathfrak{h} = \bigcap\{\, ker\lambda_i \mid i \in \mathcal{S} \,\}$. Also, for each $i \in \mathcal{S}$, let m_i be the minimal degree in x_i of nonzero elements of $P \cap U(\mathfrak{g}_i) \setminus (P \cap U(\mathfrak{g}_{i-1}))U(\mathfrak{g}_i)$. Suppose Q and P are distinct prime ideals in $U(\mathfrak{g})$.*

(a) $P \rightsquigarrow P$ if and only if $P \neq 0$.

(b) If $Q \rightsquigarrow P$, then $P = \tau_{m_i\lambda_i}(Q)$ for some $i \in \mathcal{S}$ with $\lambda_i \neq 0$. Conversely, if $i \in S$ is maximal with respect to $\tau_{\lambda_i}(P) \neq P$, then $\tau_{-m_i\lambda_i}(P) \rightsquigarrow P$.

(c) P is classically localizable if and only if $P = (P \cap U(\mathfrak{h}))U(\mathfrak{g})$. ∎

Borho has shown that if $\mathfrak{g}$ is a complex semisimple Lie algebra and P is a primitive ideal of $U(\mathfrak{g})$, then the clique of P is the set of primitive ideals Q of $U(\mathfrak{g})$ with the same central character and the same associated variety as P.

Appendix B. Group rings and group-graded rings

Recall that a group G is *polycyclic-by-finite* if it has a finite chain of subgroups from 1 to G such that each subgroup is normal in the one above it and the corresponding factor group is either cyclic or finite. If there exists such a chain in which each factor group is cyclic, we say G is *polycyclic*. Clearly a finitely generated Abelian group is polycyclic, and it is not hard to see that any finitely generated nilpotent group is polycyclic. A mild generalization of the Hilbert Basis Theorem states that if G is a polycyclic-by-finite group and R is a right Noetherian ring, then the group ring RG is right Noetherian. (This is one of the reasons we stick to polycyclic-by-finite groups.)

We say a polycyclic group H is *orbitally sound* if for any subgroup K of H with only finitely many conjugate subgroups in H, the intersection of all those conjugates has finite index in K. It is known that any polycyclic-by-finite group G contains an orbitally sound polycyclic subgroup H of finite index. Moreover, if R is a commutative Noetherian ring, it is known that the group ring RH is AR-separated. Thus RH satisfies the strong second layer condition. By the "strong" version of Theorem 7.8, it follows that RG satisfies the strong second layer condition. Brown has shown that if X is a clique in RG, then $\sup\{\,\text{uniform dim}\, R/P \mid P \in X\,\} < \infty$. Thus the localization theory works in RG given the appropriate cardinality assumptions.

In Section 4, we showed that every prime ideal in the Ore extension $R[\theta;\delta]$, where R is a commutative Noetherian ring containing $\mathbb{Q}$ and δ is a derivation on R, satisfies the second layer condition. A similar proof shows that this is also true in the Ore extensions $R[x;\phi]$ and $R[x,x^{-1};\phi]$ when R is a commutative Noetherian ring and ϕ is an automorphism of R. We now outline a proof of a result which greatly generalizes these results and the second layer condition result for groups rings stated in the previous paragraph. Analogous results hold for the PBW extensions discussed in Appendix A. For more details see the papers of Bell listed in Section 8.

Suppose G is a group (with identity 1) and S and R are rings. We say S is a *G-graded ring with base ring R* if we can write $S = \oplus_{g\in G} S(g)$ for some additive subgroups $S(g)$ of S with the property that $S(1) = R$ and that $S(g)S(h) \subseteq S(gh)$ for all $g, h \in G$. If in fact $S(g)S(h) = S(gh)$ for all $g, h \in G$, we say S is *strongly G-graded*.

The skew Laurent ring $S = R[x, x^{-1}; \phi]$ is an example of a strongly $\mathbb{Z}$-graded ring with base ring R. (Here $S(n) = Rx^n$.) Clearly the ordinary group ring $S = RG$ is a strongly G-graded ring with base ring R.

Using ideas like those in the proof of the Hilbert Basis Theorem, one can show that if G is a polycyclic-by-finite group, R is a right Noetherian ring, and S is a strongly G-graded ring with base ring R, then S is right Noetherian. Our main result is the following.

Theorem B.1. *Let S be a strongly G-graded ring with base ring R, where R is a Noetherian ring and G is a polycyclic-by-finite group. Then every prime ideal of S has the second layer condition if either R is commutative, R is p.i., R is Artinian, R is simple, or every ideal in R is principal as a right and a left ideal.*

To prove this, we first need to prove the following lemma.

Lemma B.2. *Let G be a polycyclic-by-finite group and let S be a strongly G-graded ring with right Noetherian base ring R. There does not exist a finitely generated tame*

uniform right S-module M with $\operatorname{ann}_S(M) = Q \subset P = \operatorname{ass}_S(M)$ where Q and P are prime ideals of S with $Q \cap R = P \cap R$.

Proof. We prove the result in the lemma inductively. Since G is polycyclic-by-finite, it is known that G has a series of normal subgroups $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n \subseteq G$ such that G/G_n is finite and G_i/G_{i-1} is free Abelian of finite rank for $i = 1, \ldots, n$. We assume that we have a counterexample to the lemma, but that the lemma is true for any S', G', R' such that G' has a series as just described with a smaller value of n.

If $n = 0$, then G is finite, so there cannot be prime ideals Q, P in S with $Q \subset P$ and $Q \cap R = P \cap R$. Thus we may assume $n \geq 1$.

It is not hard to see that $(Q \cap R)S = S(Q \cap R)$ is an ideal of S, and that $S/(Q \cap R)S$ is a strongly G-graded ring with base ring $R/Q \cap R$. Thus we may pass to $S/(Q \cap R)S$, and so assume $Q \cap R = P \cap R = 0$. One can show that if N is the prime radical of R, then $NS = SN$, and so NS is nilpotent. This implies $NS \subseteq Q$, and so $N = 0$. We conclude that R is semiprime.

Thus the set C of regular elements of R is a right Ore set in R and the ring RC^{-1} is semisimple Artinian. It can be shown that C is a right Ore set in S, and of course it is disjoint from Q and P. As noted before, the module MC^{-1} has annihilator QC^{-1} and assassinator PC^{-1} in SC^{-1}, and still gives a counterexample to the second layer condition. The ring SC^{-1} is a strongly G-graded ring with base ring RC^{-1}. The upshot of this is that we may assume R is semisimple Artinian.

For any subgroup H of G, $S(H) = \oplus_{g \in H} S(g)$ is a subring of S which is strongly H-graded, with base ring R, and since H is polycyclic-by-finite, $S(H)$ is right Noetherian. If $H \lhd G$, we can regard S as a strongly G/H-graded ring with base ring $S(H)$. In particular, if we define R_i to be $S(G_i)$, then S is a strongly G/G_1-graded ring with right Noetherian base ring R_1, so the induction hypothesis implies $Q \cap R_1 \subset P \cap R_1$.

The group G_1 is Abelian and (regarding S as a strongly G/G_1-graded ring with base ring R_1) $Q \cap R_1$ and $P \cap R_1$ are "G/G_1-invariant" ideals of R_1 in a sense we leave the reader to define. Since R is semisimple Artinian, it is a finite product of rings without any nontrivial "G_1-invariant" ideals. Using a minimal length argument, one can show that $P \cap R_1/Q \cap R_1$ contains a nonzero central element of $R_1/Q \cap R_1$. (Assuming R has no nontrivial "G_1-invariant" ideals, we can find a nonzero element of $(P \cap R_1) \setminus (Q \cap R_1)$ of minimal length having 1 as its coefficient in $S(1)$: such an element is central in $R_1/Q \cap R_1$.)

With this information, we no longer need the ring R. Note that $(Q \cap R_1)S = S(Q \cap R_1)$, so we may pass to $S/(Q \cap R_1)S$, which is strongly G/G_1-graded with base ring $R_1/Q \cap R_1$. We may pass to these factor rings, and so assume $Q \cap R_1 = 0$. One can now show that the ideal I of R_1 generated by the central elements of $P \cap R_1$ is nonzero and is "G/G_1-invariant". Thus $IS = SI$ is an ideal of S. Using Lemma 3.1, the Rees ring $S^*(IS)$, and the Hilbert Basis Theorem, one sees that IS has the AR property in S. Also, $IS \subseteq P$, but $IS \not\subseteq Q$, so $Q \subset Q + IS \subseteq P$ and $(Q + IS)/Q$ has the AR property in S/Q. This contradicts the existence of a counterexample to the second layer condition, as we saw in the proof of Theorem 4.7 or the remarks before Proposition 3.2. ∎

Proof of the theorem. Suppose we have a finitely generated tame uniform right S-module M with $\operatorname{ann}_S(M) = Q \subset P = \operatorname{ass}_S(M)$ for some prime ideals Q, P in S. We saw in the last lemma that $Q \cap R = P \cap R$ is not possible. As before, we may assume that

$Q \cap R = 0$ and so $P \cap R \neq 0$. If R is either Artinian or simple, the existence of such primes is impossible.

In each of the other cases listed in the hypothesis, there is a nonzero "G-invariant" ideal $I \subseteq P \cap R$ of R such that the Rees ring $R^*(I)$ is Noetherian. (If R is commutative or I is principal on both sides, we may take $I = P \cap R$. If R is p.i., we may take I to be the ideal of R generated by the central elements of $P \cap R$.) As in the proof of the lemma, this implies that $IS = SI$ has the AR property in S, contradicting the existence of such a counterexample M. ∎

Using the ideas that give us central elements and ideals with the AR property in the above proofs (along with localization), one can prove the following.

Theorem B.3. *Let R be a Noetherian ring, G a polycyclic-by-finite group, S a strongly G-graded ring with base ring R, and P a prime ideal of S. If either (i) G is a finitely generated nilpotent group and $P \cap R = 0$ or (ii) R is commutative and $P = IS = SI$ for an ideal I of R, then P is classically localizable.* ∎

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